Model-theoretic Galois theory

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What is Galois theory?

- In the classical, algebraic sense: the classification of intermediate field extensions in a field-theoretic algebraic closure according to the structure of a profinite automorphism group.
- In the classical, model-theoretic sense: the classification of definably closed subsets of a model-theoretic algebraic closure according to the structure of a profinite automorphism group. (Poizat, 1983)
- In the modern, algebraic sense: the classification of locally constant sheaves on a site according to the structure of a category of finite G-sets, where G is some flavor of fundamental group. (Grothendieck, SGA)
- In the modern, model-theoretic sense: Lascar groups, (bounded) hyperimaginaries, generalized imaginaries and definable groupoids, internal covers?

Elimination of imaginaries

Recall:

- A first-order theory T (uniformly) *eliminates imaginaries* if every 0-definable equivalence relation E arises as the kernel relation $\models \ker(f)(a,b) \iff \models f(a) = f(b)$ of a 0-definable function f.
- T *codes definable sets* if for every $X = \varphi(\mathbb{M}, b)$, there exists a tuple c and a formula $\psi(x,c)$ such that $X = \psi(\mathbb{M}, c)$, with c unique. We usually suppress ψ and say that c *codes* X.

- EI always implies coding; the converse holds with some mild conditions on T (satisfied by ACF).
- In the monster $\mathbb{M}$, codes are characterized up to interdefinability by the following property: c codes X precisely when

  $\mathrm{Aut}(\mathbb{M})$ fixes X setwise $\iff$ $\mathrm{Aut}(\mathbb{M})$ fixes c pointwise.

Recovering the Galois correspondence

Definition.

- Let T be a first-order theory, and $\mathbb{M} \models T$ a monster. If $A \subseteq \mathbb{M}$ is a small parameter set, $\mathrm{Aut}(\mathbb{M}/A)$ acts on $\mathrm{acl}(A)$. The image of the action

$$\mathrm{Aut}(\mathbb{M}/A) \to \mathrm{Sym}(\mathrm{acl}(A)/A)$$

is the *absolute Galois group of A*, and we denote it by $\mathrm{G}(\mathrm{acl}(A)/A)$.

- $\mathrm{Aut}(\mathbb{M}/A)$ similarly induces a finite $G(B/A)$ for any finite A-definable set B, and in fact we have (as one might hope) $G(\mathrm{acl}(A)/A) \simeq \varprojlim G(B/A)$, so $G(\mathrm{acl}(A)/A)$ is naturally a profinite group.

Recovering the Galois correspondence

Theorem. (Poizat) Let T code definable sets, $\mathbb{M} \models T$ be a monster, and $A \subseteq \mathbb{M}$ small. Then

$$\begin{tikzcd} \mathbf{Sub}_{\text{pro-closed}}\left(G\left({}^{\mathrm{acl}(A)}/_{A} \right) \right) \arrow[r, bend left, "\mathrm{Fix}(-)"] & \mathbf{Sub}_{\text{dcl-closed}}\left({}^{\mathrm{acl}(A)}/_{A} \right) \arrow[l, bend left, "G(\mathrm{acl}(A)/-)"] \end{tikzcd}$$

is an inclusion-reversing bijective correspondence between the subgroups of $G(\mathrm{acl}(A)/A)$ closed in the profinite topology and definably-closed intermediate extensions of A, where $\mathrm{Fix}(-)$ is given by

$$H \mapsto \{ b \in \mathrm{acl}(A) \mid \sigma(b) = b, \forall h \in H \}$$

and $G(\mathrm{acl}(A)/-)$ by

$$B \mapsto \{ \sigma \in G(\mathrm{acl}(A)/A) \mid h(b) = b, \forall b \in B \}.$$

Recovering the Galois correspondence

Proof sketch.

1. Use coding of definable sets and the fact that algebraic elements correspond to certain strong types to establish a bijection $G(\mathrm{SF}(A)/A) \simeq G(\mathrm{acl}(A)/A)$, where $\mathrm{SF}(A)$ is the space of strong types over A.

2. Using the fact that $\mathrm{stp}(a/A) = \mathrm{stp}(b/A) \iff a \sim_E b$ for every A-definable finite equivalence relation E, establish that $G(\mathrm{SF}(A)/A) \simeq$ inverse limit of $G(E/A)$, where $G(E/A)$ is the image of the action of $\mathrm{Aut}(\mathbb{M}/A)$ on E-classes.

3. To every finite A-definable equivalence relation E with classes $C_1, \ldots, C_n$ and every subgroup $\Gamma \subseteq G(E/A)$, associate a definable Γ-invariant relation $r_\Gamma(x_1, \ldots, x_n) \iff \bigvee_{\sigma \in \Gamma} \bigwedge_{i \leqslant n} x_i \in \sigma(C_i)$. Denote the subgroup of such $\sigma \in G(\mathrm{SF}(A)/A)$ as $\mathrm{Stab}(r_\Gamma)$.

Recovering the Galois correspondence

4. Since cylinder sets (hence stabilizers) are clopen in the profinite topology, $G(\mathrm{acl}(A)/B)$ is closed. By homogeneity of the monster, $\mathrm{Fix}(H)$ is definably closed. So the maps are well-defined.

5. That $\mathrm{Fix}(-)$ is left-inverse to $G(\mathrm{acl}(A)/-)$ is also a direct consequence of homogeneity in the monster.

6. To see that $G(\mathrm{acl}(A)/-)$ is left-inverse to $\mathrm{Fix}(-)$, write a closed subgroup H as an intersection of open subgroups, each of the form $\mathrm{Stab}(r_{\Gamma_i})$. Since each r_{Γ_i} as a set has only finitely many A-conjugates, any of its codes c_i is A-algebraic. Let B be the set of all codes for all r_{Γ_i}. Since we're in a monster, this is definably closed, and we see that $G(\mathrm{acl}(A)/B) = H$. By (5), $\mathrm{Fix}(H) = B$, so $G(\mathrm{acl}(A)/-)$ left-inverts $\mathrm{Fix}(-)$. ✓

Recovering the Galois correspondence

- **Fact.** ACF eliminates imaginaries.
- **Corollary.** The same idea (with considerably less effort) works to recover the correspondence for finitely-generated definably-closed normal extensions, where "normal over A" means "closed under taking $\mathrm{Aut}(\mathbb{M}/A)$-orbits".
- **Corollary.** Classical Galois theory, at least in characteristic 0.
- In characteristic p, due to the definability of Frobenius, definable closures are perfect hulls, and model-theoretic algebraic closures are field-theoretic algebraic closures of perfect hulls. So while our absolute Galois groups coincide with classical absolute Galois groups, they require the ground field to be perfect.

Hyperimaginaries, ultraimaginaries

Definition.

- A subset $X \subseteq \mathbb{M}$ is *type-definable* if it is a possibly-infinite conjunction of definable sets.
- A *hyperimaginary* is an E-class (of possibly infinite tuples) where E is a type-definable equivalence relation.
- An *ultraimaginary* is an E-class (of possibly infinite tuples) where E is an $\mathrm{Aut}(\mathbb{M})$-invariant equivalence relation.
- If α is a flavor of imaginary, α is said to be *bounded* if its orbit under $\mathrm{Aut}(\mathbb{M})$ is small (equivalently, if its parent equivalence relation E has only a small number of classes.) α is said to be (in)*finitary* if its parent equivalence relation E relates (in)finite tuples.

What is the Lascar group?

- Rather than classifying $\mathrm{dcl}^{\mathrm{eq}}$-closed parameter sets in $\mathbb{M}^{\mathrm{eq}}$, the closed subgroups of the Lascar group will be in Galois correspondence with finitary bounded hyperimaginaries (so, certain small quotients of $\mathbb{M}^{\mathrm{eq}}$ instead of certain small subsets.)
- But let's actually define this thing.
- First, we need to define: *the group of Lascar strong automorphisms* of $\mathbb{M}$ over a small parameter set A, which is the normal subgroup $\mathrm{Autf}(\mathbb{M}/A)$ of $\mathrm{Aut}(\mathbb{M}/A)$ generated by
$$\bigcup_{A \subseteq \mathcal{M} \prec \mathbb{M}} \mathrm{Aut}(\mathbb{M}/\mathcal{M}).$$
- This fixes bounded ultraimaginaries (in fact is the stabilizer of bounded ultraimaginaries).

What is the Lascar group?

- **Definition.** The *Lascar group* of $T = \mathrm{Th}(\mathbb{M})$ over a small parameter set A is the quotient

$$\mathrm{Aut}(\mathbb{M}/A) \Big/ \mathrm{Autf}(\mathbb{M}/A),$$

  denoted $\mathrm{Gal}_L(T/A)$.
- This acts faithfully on bounded ultraimaginaries.
- Topologize it as follows: for each E a parent equivalence relation of a bounded ultraimaginary, define a topology on the quotient (for some appropriate power of $\mathbb{M}$) $\mathbb{M}/E$ by setting $X \subseteq \mathbb{M}/E$ closed $\iff$ $\pi_E^{-1}(X)$ is an intersection of definable sets. $\mathrm{Gal}_L(T)$ acts on each $\mathbb{M}/E$, and we take its topology to be the coarsest one making all the actions continuous.

How does $\mathrm{Gal}_L(T/A)$ relate to the absolute Galois group?

- When $\mathrm{acl}(A) \prec \mathbb{M}$, $\mathrm{Autf}(\mathbb{M}/A) = \mathrm{Aut}(\mathbb{M}/\mathrm{acl}(A))$, so that $\mathrm{Gal}_L(T/A) \simeq G(\mathrm{acl}(A)/A)$.
- This holds, for example, in ACF.
- More generally, let $\mathrm{Gal}^0_L(T/A)$ be the connected component of the identity. This is isomorphic to $\mathrm{Aut}(\mathbb{M}/\mathrm{acl}^{\mathrm{eq}}(A))$ and we always have a short exact sequence

$$1 \to \mathrm{Gal}^0_L(T/A) \to \mathrm{Gal}_L(T/A) \to \mathrm{G}(\mathrm{acl}^{\mathrm{eq}}(A)/A) \to 1.$$

  and if T is stable, $\mathrm{Gal}^0_L(T/A) = 1$.

A hyperimaginary Galois theory

Theorem. (Lascar, Pillay) Taking fixpoints and taking stabilizers yields a Galois correspondence between the closed subgroups of $\mathrm{Gal}_L(T/A)$ and definably-closed sets of hyperimaginaries over A.

Proof sketch.

- Show that $G \subseteq \mathrm{Gal}_L(\mathbb{M}/A)$ is closed if and only if $G = \mathrm{Stab}(e)$ for some bounded hyperimaginary e.
- Show that bounded hyperimaginaries are equivalent to sequences of finitary bounded hyperimaginaries.
- This last step requires some structure theory of compact Hausdorff topological groups, namely that we can realize such groups as the projective limit of compact Lie groups.

Grothendieck's Galois theory

- **Definition.** (Grothendieck, SGA IV) A *Galois category* is a small Boolean pretopos $\mathbf{C}$ with an exact isomorphism-reflecting functor (the fiber functor)

$$\mathbf{C} \xrightarrow{F} \mathbf{FinSet}.$$

- The content of Grothendieck's Galois theory establishes an equivalence $\mathbf{C} \simeq G\text{-}\mathbf{Set}$, where G is the automorphism group of the fiber functor, and G-$\mathbf{Set}$ is the category of continuous G-sets. When $\mathbf{C}$ is the category of finite etale coverings of a scheme X, G is the *etale fundamental group* $\pi_1(X)$. When $X = \mathrm{Spec}(K)$, $\pi_1(X) \simeq \mathrm{Gal}(K^{\mathrm{sep}}/K)$.
- In fact, we can see that if T eliminates imaginaries, the category $\mathbf{FinDef}(T/A)$ of finite A-definable sets equipped with the forgetful functor to $\mathbf{FinSet}$ is a Galois category.

Some wild speculation

- **Fact.** Let X be an algebraic variety over a field K. Let $X_{K^{\mathrm{sep}}}$ be the base change of X along $\mathrm{Spec}(K^{\mathrm{sep}}) \to \mathrm{Spec}(K)$. If both X and $X_{K^{\mathrm{sep}}}$ are connected, we have an exact sequence

$$1 \to \pi_1(X_{K^{\mathrm{sep}}}) \to \pi_1(X) \to \pi_1(K) \to 1.$$

- **Question.** Can we construe $\mathrm{Gal}_L(T/A)$ or $\mathrm{Gal}^0_L(T/A)$ as "global" fundamental groups? What definable structures do they classify?
- Makkai observed that if T eliminates imaginaries, $\mathbf{Def}(T)$ is already equivalent to a Boolean pretopos. So, **Question.** If T has EI, what conditions are needed on small subpretoposes $\mathbf{C} \hookrightarrow \mathbf{Def}(T)$ and the process of taking codes to make $\mathbf{C} \xrightarrow{\mathrm{code}(-)} \mathbf{FinSet}$ a Galois category?

Interpretations and stable-embeddedness

- **Definition.** Let T_1 and T_2 be $\mathcal{L}_1$- and $\mathcal{L}_2$-theories, both complete. A *definition* $\pi : T_1 \to T_2$ of T_1 in T_2 is an assignment:
    - of an $\mathcal{L}_2$-formula $\pi(S)$ to each sort S of $\mathcal{L}_1$,
    - of an $\mathcal{L}_2$-formula $\pi(R)$, appropriately sorted, to each nonlogical symbol R of $\mathcal{L}_1$, such that
    - for any $M \models T_2$, the $\mathcal{L}_1$-structure $\pi^*(M)$—interpreting each S a sort of $\mathcal{L}_1$ as $\pi(S)$ and each nonlogical symbol R of $\mathcal{L}_1$ as $\pi(R)$—is a model of T_1.

- **Definition.** An *interpretation* $T_1 \to T_2$ is a definition of T_1 in $(T_2)^{\text{eq}}$.

- **Definition.** An interpretation $\pi : T_1 \to T_2$ is *stably-embedded* if for any model $M \models T_2$, any definable subset of $\pi^*(M)$ which is M-definable with an $\mathcal{L}_2$-formula is also $\pi^*(M)$-definable with an $\mathcal{L}_1$-formula.

Internal covers

- **Definition.** An *internal cover* of a theory T_0 is a stably-embedded intepretation $\pi : T \to T_0$ which admits a stably-embedded section $\iota : T_0 \to T$.

- **Definition.** The (set of) *internality parameters* of an internal cover $T \xrightarrow{\pi} T_0$ is a set of parameters A such that for any $M \models T_0$, $\mathrm{dcl}(\pi^* \circ \iota^*(M) \cup A) = M$.

- To any internal cover of $T \xrightarrow{\pi} T_0$ we can associate a *definable* Galois theory: a pro-definable group $\mathbf{G}$ in T with a pro-definable action $\mathbf{G} \curvearrowright Q$ for every definable Q in T.

- Furthermore, given a set of internality parameters A (with $A_0 = \iota \circ \pi(A)$), there is a Galois correspondence

$$\{\text{pro-A-definable } H \underset{\text{closed}}{\subseteq} \mathbf{G}\} \leftrightarrows \{\text{dcl-closed } A_0 \subseteq B \subseteq A\}.$$

The Tannakian formalism

- Tannaka duality generalizes Grothendieck's Galois theory, and centers around the *reconstruction* of algebraic symmetry objects from (a fiber functor on) a symmetric monoidal category $\mathbf{C}$ of representations of that object (which forgets the representation data).
- Canonical example: reconstruction of an absolute Galois group of a field k from (the geometric points functor $\text{Hom}(-, k^{\text{sep}}/k)$ on) the category $\mathbf{C}$ of finite etale k-algebras with tensor product.
- **Definition.** A *neutral Tannakian category over a field k* is the data $(\mathbf{C}, F)$, where $\mathbf{C}$ is a rigid symmetric monoidal category $(\mathbf{C}, \otimes)$ such that the endomorphism algebra of $\mathbf{C}$'s terminal object is k, and F is an exact functor $\omega : \mathbf{C} \to \mathbf{Vec}_k$ which preserves tensor products.

The Tannakian formalism

- Kamensky (2010) uses internal covers to give a slightly-weaker model-theoretic account of Tannaka duality for such categories when k has characteristic 0 by assigning to each $\mathbf{C}$ a theory $T_{\mathbf{C}}$ whose models are fiber functors.
- This almost recovers an affine group scheme as the automorphism group of the fiber functor ω.
- The above method, however, is enough to recover Grothendieck's Galois theory in its full generality.

Homology and cohomology

- I'll wrap up by briefly mentioning recent work relating model theory and (co)homology.
- Goodrick, Kim, Kim, Kolesnikov and Lee have begun fleshing out the theory of homology groups of types in rosy theories.
- Maybe relevant to an earlier wild speculation: Lee has shown that in certain cases the first homology group of a strong type over an algebraically closed set A arises as the abelianization of the Lascar group over A, similar to a Hurewicz map.
- Recently, Sustretov has classified classes in the second cohomology group of a model-theoretic absolute Galois group in terms of Morita-equivalent definable groupoids.

Thank you!