Definable categories

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Abstract

In these notes we study some aspects of internal category theory in the syntactic category $\mathbf{Def}(T)$ of a first-order theory T; that is, of categories definable in T.

Disclaimer

These are extremely rough notes, kept for the purpose of writing down definitions and new and partial results from work-in-progress. As a result, notation is inconsistent and I have not yet written references or cross-references. Often I resort to using elements and so much of the material here has not been developed in the full generality it ought to be, though everything works in $\mathbf{Def}(T)$ (implicitly taking points in a monster model), but we should be able to formulate and prove the results wholly diagrammatically, maybe modulo a Grothendieck topology.

Comments, questions, and notes on typos or more serious errors are welcome.

A note on notation: when $\mathbf{C}$ is an internal category, we write C_0 and C_1 for the object-of-objects and the object-of-morphisms. We also sometimes write $(\mathbf{C})_0$ and $(\mathbf{C})_1$ for the same thing. Also, the internalization of small-completeness and small-continuity for categories and functors is just internal completeness and continuity: all limits of diagrams internally indexed by internal categories exist (resp. are preserved).

Introduction

Just as we can study groups definable in some first-order theory T, which are just group objects in the category $\mathbf{Def}(T)$ of definable sets (and definable functions between them) in T, we can study categories definable in T, i.e. category objects, or *internal categories*, in $\mathbf{Def}(T)$.

Internal categories arise naturally elsewhere in mathematics (e.g. internal categories are internal groupoids are crossed modules in $\mathbf{Grp}$, and internal congruences (e.g. equivalence relations in $\mathbf{Set}$, ideals in $\mathbf{CRing}$ and normal subgroups in $\mathbf{Grp}$) can be identified as internal groupoids.) So it is natural to ask what internal categories in $\mathbf{Def}(T)$ have to say about T. Conversely, it's also natural to ask how T influences how much category theory we can recover from just the categories definable in T.

In particular, we look at definable adjoint pairs of definable functors, and recover the general adjoint functor theorem internal to $\mathbf{Def}(T)$, modulo a definable Skolem function. Motivated by the desire to rid ourselves of this last requirement, we turn to the internalizations to $\mathbf{Def}(T)$ of anafunctors instead, which were introduced by Makkai to generalize functors and perform limit constructions in settings without choice, and we prove a general adjoint functor theorem for definable anafunctors.

Along the way, we note that T having definable Skolem functions is precisely the external axiom of choice for $\mathbf{Def}(T)$ equipped with the regular coverage: that all definable surjections admit a definable section. We then internalize to $\mathbf{Def}(T)$ the fact that the external axiom of choice for **Set** is equivalent to being able to upgrade fully faithful essentially surjective functors between small categories to full equivalences of categories. This gives a characterization of having definable Skolem functions in terms of definable functors.

We also observe that saturated anaequivalences between definable categories provide a generalization of Morita equivalence between definable groupoids; this has recently been applied to characterize generalized imaginary sorts. We then show that two definable categories are saturated anaequivalent if and only if there is a bibundle between them, and generalize appropriate parts of the theory relating definable groupoids and internal covers.

Contents

1 Categories as two-sorted first-order structures

Consider the language $\mathcal{L}_{\text{cat}}$ with two sorts $\{\mathrm{Ob}, \mathrm{Mor}\}$, function symbols dom and cod (sometimes written s and t for *source* and *target*) from Mor to Ob, a binary partial *composition function* $\circ$ whose graph relation $\circ(-,-,-) \subseteq \mathrm{Mor} \times \mathrm{Mor} \times \mathrm{Mor}$ we include as a symbol in the language.

Since a definable category $\mathbf{C}$ in an arbitrary first-order theory T is a (strict) interpretation of a pure category (i.e. an $\mathcal{L}_{\text{cat}}$-structure satisfying the category axioms) in T, we can see what definable sets come for free with $\mathbf{C}$ in T by looking at what is definable in a pure category.

Definition 1.1. A *category* is an $\mathcal{L}_{\text{cat}}$-structure satisfying the following axioms, which state that composition is only defined on arrows with compatible domains and codomains, that composition is associative, and that there are elements which are simultaneously left- and right-identities for composition:

(i) $\forall f\ \forall g\ \big[\mathrm{cod}(f) \neq \mathrm{dom}(f) \to \forall h\ (\neg \circ (f, g, h))\big]$.

(ii) $\forall f\ \forall g\ \forall h\ \big[\circ (f, g, h) \leftrightarrow [\forall h'\ \ (\circ(f, g, h') \to h = h') \wedge \mathrm{dom}(h) = \mathrm{dom}(g) \wedge \mathrm{cod}(h) = \mathrm{cod}(f)]\big]$.

(iii) $\forall f\ \forall g\ \forall h\ \circ (f, \circ(g, h)) = \circ(\circ(f, g), h)$.

(iv) $\forall a \in \mathrm{Ob}\ \exists e_a \in \mathrm{Mor}\ \big(\mathrm{cod}(e_a) = \mathrm{dom}(e_a) = a\ \wedge \forall f\ (\mathrm{dom}(f) = a \to \circ(f, e_a, f) \wedge \mathrm{cod}(f) = a \to (e_a, f, f))\big)$.

As usual, we write $\circ$ infix whenever it is defined as a function, and write $X \xrightarrow{f} Y$ to denote that f has domain X and codomain Y. When we need to write composition as a prefix operator, we use "c" instead of "$\circ$". Identity maps are necessarily unique, so the function $\mathrm{id} : \mathrm{Ob} \to \mathrm{Mor}$ taking objects to their identity maps is 0-definable.

Definition 1.2. An object $c \in \mathbf{C}$ a definable category is *definable up to isomorphism* if some subset of the isomorphism class of c in $\mathbf{C}$ is definable.

The following examples are all definable up to isomorphism.

Example 1.3. Initial and terminal objects: consider

$$\mathsf{isInitial}(x) \overset{\text{df}}{\equiv} \forall y \in \mathrm{Ob}\ \exists! f : x \to y,$$

and dually for terminal objects. These are definable up to isomorphism.

Example 1.4. Slice and co-slice categories: fix a category $\mathbf{C}$ and a base $b \in \mathrm{Ob}$. For the objects of the slice category, take the definable set

$$\mathrm{Ob}(\mathbf{C}/b) \overset{\text{df}}{=} \{f \in \mathrm{Mor} \,|\, \mathrm{cod}(f) = b\}.$$

For morphisms, take

$$\mathrm{Mor}(\mathbf{C}/b) \overset{\mathrm{df}}{=} \{f \in \mathrm{Mor} \,\big|\, f : \mathrm{dom}(g) \to \mathrm{dom}(h) \text{ for } g, h \in \mathrm{Ob}(\mathbf{C}/b) \text{ such that } h \circ f = g\}.$$

Remark 1.5. Note that in the same way algebraic groups are group objects in the category of algebraic varieties, this defines a *category object*, i.e. an internal category in $\mathbf{Def}(\mathbf{Th}(\mathbf{C}))$.

Example 1.6. Limits and colimits of arbitrary finite diagrams: if $\mathbf{D}$ is a diagram with finitely many objects $d_1, \ldots, d_n$ with finitely many morphisms between them, a limit to $\mathbf{D}$ is just a tuple $(x, \pi_1, \ldots, \pi_m)$ where x is an object and the π_i are maps $x \to d_i$ such that:

- **(i)** whenever $f : d_i \to d_j \in \mathbf{D}$, $f \circ \pi_i = \pi_j$ and
- **(ii)** whenever we have another tuple $(y, \pi'_1, \ldots, \pi'_n)$ satisfying the above conditions, there exists a unique map $y \xrightarrow{u} x$ such that $\pi'_i = \pi_i \circ u$ for each $1 \leqslant i \leqslant n$.

(Alternately, we can just modify our realization of slice and co-slice categories as definable categories in $\mathbf{C}$ to realize cone and co-cone categories as definable categories in $\mathbf{C}$. Then limits and colimits are just the terminal and initial objects in those definable categories, hence definable.)

Example 1.7. Limits and colimits of arbitrary definable diagrams construed as definable subcategories: in the same spirit as the above example, except instead of capturing the entire diagram in a sentence, as is possible when $\mathbf{D}$ is finite, we only need check that certrain subtriangles of our diagram commute, and as long as the legs of the triangles belong to something we can safely quantify over, it doesn't matter if there are infinitely many things to check.

Definition 1.8. Let $\mathbf{A}, \mathbf{B}$ be two definable categories in $\mathbf{C}$. A *definable functor* $F : \mathbf{A} \to \mathbf{B}$ comprises definable maps $F_0 : A_0 \to B_0$ and $F_1 : A_1 \to B_1$ which behave like the data of a usual functor with respect to the internal composition, domain, and codomain maps of either internal category. If F and G are two definable functors, a *definable natural transformation* $\eta : F \to G$ is a definable function $A_0 \to B_1$ such that the following diagrams commute:

$$\begin{tikzcd} B_1 \arrow[r, "(s,t)"] & B_0 \times B_0 \\ A_0 \arrow[u, "\eta"] \arrow[ur, "(F_0,G_0)"'] & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} B_1 \times_{s,B_0,t} B_1 \arrow[rr, "c"] & & B_1 \\ A_1 \arrow[u, "(\eta\circ t, F_1)"] \arrow[r, "(G_1, \eta\circ s)"'] & B_1 \times_{s,B_0,t} B_1. \arrow[ur, "c"'] & \end{tikzcd}$$

Example 1.9. Change-of-base functors between slice-categories: if pullbacks exist, this is clear.

Example 1.10. Epimorphisms, monomorphisms, and subobject classifiers: to be a monomorphism $f\colon X \to Y$ means that the definable map $(f \circ -) : \mathrm{Hom}(-, X) \to \mathrm{Hom}(-, Y) = f \circ - : \mathrm{cod}^{-1}(X) \to \mathrm{cod}^{-1}(Y)$ is injective. To be an epimorphism $f\colon X \to Y$ means that the definable map $- \circ f\colon \mathrm{Hom}(-, Y) \to \mathrm{Hom}(-, X)$ is injective. To have a subobject classifier is to say that there exists a terminal object $\mathbf{1}$ and a monomorphism $\mathrm{true} : \mathbf{1} \to \Omega$ for some object

Ω such that for every object X and monomorphism $S \hookrightarrow X$, there is a unique map $X \to \Omega$ such that $S \hookrightarrow X$ is the pullback of true along the map $X \to \Omega$, which is first-order.

Example 1.11. Power objects of a fixed object X: "there exists an object Ω^X and a monomorphism $\in_X \hookrightarrow X \times \Omega^X$ such that for any other object Y and every monomorphism $S \hookrightarrow X \times Y$ there is a unique classifying map $X \times Y \to X \times \Omega^X$ such that $S \hookrightarrow X \times Y$ is the pullback of $\in_X \hookrightarrow X \times \Omega^X$ along that classifying map."

Exercise 1.12. (For the masochistic.) Write out the last four examples explicitly.

Remark 1.13. I don't think arbitrary limits and colimits are definable. So, to-do: construct two categories equivalent as $\mathcal{L}_{\text{cat}}$-structures, but which realize some limit or colimit differently.

2 Definable adjunctions

In this section we study pairs of definable adjoint functors between definable categories.

Proposition 2.0.1. *Let $\mathbf{D}_1$ and $\mathbf{D}_2$ be definable categories in T a first-order theory. Let $F : \mathbf{D}_1 \to \mathbf{D}_2$ be a definable functor which is left-adjoint to G. Suppose the family of hom-set bijections*

$$\{\phi_{X,Y} \colon \operatorname{Hom}_{\mathbf{D}_2}(FX, Y) \xrightarrow{\sim} \operatorname{Hom}_{\mathbf{D}_1}(X, GY)\}_{X \in \mathbf{D}_1, Y \in \mathbf{D}_2}$$

is definable as a function

$$\coprod_{X \in \mathbf{D}_1, Y \in \mathbf{D}_2} \operatorname{Hom}_{\mathbf{D}_2}(FX, Y) \longrightarrow \coprod_{X \in \mathbf{D}_1, Y \in \mathbf{D}_2} \operatorname{Hom}_{\mathbf{D}_1}(X, GY).$$

Then if $\mathbf{D}_1$ has enough projectives, $G : \mathbf{D}_2 \to \mathbf{D}_1$ is definable also.

Proof. The definability of the hom-set bijection means that what G does on objects is already definable. Consider the relation $\Gamma \subseteq \operatorname{Mor}(\mathbf{D}_2) \times \operatorname{Mor}(\mathbf{D}_1)$ by $(f, \overline{f}) \in \Gamma$ if and only if $\forall X$,

$$\begin{tikzcd}
\operatorname{Hom}_{\mathbf{D}_2}(FX, Y_1) \arrow[r, "\phi_{X,Y_1}"] \arrow[d, "f \circ -"'] & \operatorname{Hom}_{\mathbf{D}_1}(X, GY_1) \arrow[d, "\overline{f} \circ -"] \\
\operatorname{Hom}_{\mathbf{D}_2}(FX, Y_2) \arrow[r, "\phi_{X,Y_2}"'] & \operatorname{Hom}_{\mathbf{D}_1}(X, GY_2)
\end{tikzcd}$$

commutes. If $\overline{f}, \overline{f}'$ are two elements from fiber of Γ at f, then they must both satisfy $\overline{f} \circ \phi_{X,Y_1}(g) = \phi_{X,Y_2}(f \circ g) = \overline{f}' \circ \phi_{X,Y_1}(g)$ for all $g : FX \to Y_1$.

If we have a factorization
$$\begin{tikzcd} Y_1 \arrow[rr, "f"] \arrow[dr, "f_1"'] & & Y_2 \\ & Y_3 \arrow[ur, "f_2"'] & \end{tikzcd}$$
and choose from the fibers of f_1 and f_2 morphisms $\overline{f}_1 \in \Gamma(Y_1 \xrightarrow{f_1} Y_3)$ and $\overline{f}_2 \in \Gamma(Y_3 \xrightarrow{f_2} Y_2)$, then for all $X \in \mathbf{D}_1$,

$$\begin{tikzcd} \mathrm{Hom}_{\mathbf{D}_2}(FX, Y_1) \arrow[r, "\phi_{X,Y_1}"] \arrow[d, "f_1 \circ -"'] & \mathrm{Hom}_{\mathbf{D}_1}(X, GY_1) \arrow[d, "\overline{f}_1 \circ -"] \\ \mathrm{Hom}_{\mathbf{D}_2}(FX, Y_3) \arrow[r, "\phi_{X,Y_3}"] \arrow[d, "f_2 \circ -"'] & \mathrm{Hom}_{\mathbf{D}_1}(X, GY_3) \arrow[d, "\overline{f}_1 \circ -"] \\ \mathrm{Hom}_{\mathbf{D}_2}(FX, Y_2) \arrow[r, "\phi_{X,Y_2}"] & \mathrm{Hom}_{\mathbf{D}_1}(X, GY_2) \end{tikzcd}$$

commutes, so that for each $g : FX \to Y_1$,

$$\begin{aligned} \overline{f}_2 \circ \overline{f}_1 \circ \phi_{X,Y_1}(g) &= \overline{f}_2 \circ \phi_{X,Y_3}(f_1 \circ g) \\ &= \phi_{X,Y_2}(f_2 \circ f_1) \\ &= \phi_{X,Y_2}(f \circ g) \\ &= \overline{f} \circ \phi_{X,Y_1}(g). \end{aligned}$$

Similarly, $\overline{\mathrm{id}_{Y_1}} \circ \phi_{X,Y_1}(g) = \phi_{X,Y_1}(g)$. Hence, if for each Y_1 there is an X such that X admits an epimorphism to Y_1, then we can right-cancel $\phi_{X,Y_1}(g)$ in the above equations, so that Γ defines a functor right-adjoint to F. $\square$

Corollary 2.0.1. *If in the above situation G is definable instead and $\mathbf{D}_2$ has enough injectives, F is definable also.*

Proof. Immediate upon examination of the proof. $\square$

Corollary 2.0.2. *Let $F : \mathbf{D}_1 \leftrightarrows \mathbf{D}_2 : G$ be a pair of functors between two categories definable in a theory T. Then:*

- **(i)** *If $\mathbf{D}_1$ has enough projectives, F is definable, the restriction of G to objects of $\mathbf{D}_2$ is definable, and the counit ϵ is definable, then G is definable also.*
- **(ii)** *If $\mathbf{D}_2$ has enough injectives, G is definable, the restriction of F to objects of $\mathbf{D}_1$ is definable, and the unit η is definable, then F is definable also.*

Proof. $\phi_{X,Y}$ is always given by $\left(FX \xrightarrow{g} Y\right) \mapsto \left(X \xrightarrow{G(g)\circ\eta_X} GY\right)$, and its inverse is given by taking an $X \xrightarrow{f} GY$ and sending it to $\epsilon_Y \circ F(f)$ instead. $\square$

2.1 The general adjoint functor theorem

Definition 2.1. Consider the diagram of functors
$$\begin{tikzcd} \mathbf{C} \arrow[dr, "F"'] & & \mathbf{D}. \arrow[dl, "G"] \\ & \mathbf{E} & \end{tikzcd}$$
The *comma category* $(F \downarrow G)$ is given by:

Objects: (c, d, α) where $c \in \mathbf{C}, d \in \mathbf{D}, \alpha : F(c) \to G(d) \in \mathbf{E}$.

Morphisms: $\mathrm{Hom}_{(F\downarrow G)}((c_1, d_1, \alpha_1), (c_2, d_2, \alpha_2))$ is defined to be the set

$$\left\{ (\beta_1, \beta_2) \,\middle|\, \beta_1 : c_1 \to c_2, \beta_2 : d_1 \to d_2, \text{ and } \begin{tikzcd} F(c_1) \arrow[r, "F(\beta_1)"] \arrow[d, "\alpha_1"'] & F(c_2) \arrow[d, "\alpha_2"] \\ G(d_1) \arrow[r, "G(\beta_2)"] & G(d_2) \end{tikzcd} \text{ commutes.} \right\}$$

.

Definition 2.2. If $F \colon \mathbf{C} \to \mathbf{Set}$ is a $\mathbf{Set}$-valued functor on a locally small category $\mathbf{C}$, the *category of elements of* F $\int^{c\in\mathbf{C}} F(c)$ is given by:

Objects: $\{(c, x) \mid c \in \mathbf{C}, x \in F(C)\}$.

Morphisms: $\mathrm{Hom}_{\int^{c\in\mathbf{C}} F(c)}((c_1, x_1), (c_2, x_2))$ is defined to be the set

$$\{f \mid f : c_1 \to c_2 \text{ and } F(f)(x_1) = x_2.\}$$

Definition 2.3. Let $\mathbf{C} \xrightarrow{F} \mathbf{D}$ be a functor between locally small categories, and let $U \in \mathbf{D}$. The category $\mathrm{Pt}(U, F)$ of U*-points of* F is given by:

Objects: $\{\coprod_{c\in\mathbf{C}} \mathrm{Hom}_{\mathbf{D}}(U, F(c))\}$, each written (c, x) where $x : U \to F(c)$.

Morphisms: $\mathrm{Hom}_{\mathrm{Pt}(U,F)}((c_1, x_1), (c_2, x_2))$ is defined to be the set

$$\{f \mid f \colon c_1 \to c_2 \text{ and } F(f) \circ x_1 = x_2.\}$$

Remark 2.4. Note that if F lands in $\mathbf{Set}$, the category of elements of F is precisely $\mathrm{Pt}(\mathbf{1}, F)$. In particular, the category of elements of a G-set is its action groupoid.

Remark 2.5. Identifying an object d of a category $\mathbf{D}$ with the functor from the delooping of the trivial group $\mathbf{B}1 \to \mathbf{D}$ pointing at d, note that for any functor $F : \mathbf{C} \to \mathbf{D}$,

$$(d \downarrow F) = \mathrm{Pt}(d, F).$$

Lemma 2.5.1. *Let $G \colon \mathbf{D} \to \mathbf{C}$ be a functor. G admits a left adjoint F if and only if for each $c \in \mathbf{C}$ there is an initial object i_c of the comma category $(c \downarrow G)$. i_c is called the* reflection *of c along G.*

Proof. Suppose F is left adjoint to G. Let η be the unit $1_{\mathbf{C}} \to GF$. Claim: $(F(c), \eta(c) : c \to GF(c))$ is the initial object of $\mathrm{Pt}(c, G)$. To see this, let $(d, x : c \to G(d))$ be another object in $\mathrm{Pt}(c, G)$. Taking F-G transposes, as in

$$\begin{tikzcd} c \arrow[r, "\eta(c)"] \arrow[dr, "x"'] & GF(c) \arrow[d, dashed] \\ & G(d) \end{tikzcd} \longleftrightarrow \begin{tikzcd} F(c) \arrow[r, "\mathrm{id}_{F(c)}"] \arrow[dr, "\overline{x}"'] & F(c) \arrow[d, "\overline{x}"] \\ & d \end{tikzcd},$$

shows that any completion to the triangle on the left must be precisely $G(\overline{x})$, necessarily unique. On the other hand, suppose that each $\mathrm{Pt}(c, G)$ has an initial object $i_c = (d_c, \eta_c : c \to G(d_c))$. F will be defined by

$$\left(c_1 \xrightarrow{g} c_2\right) \mapsto \left(d_{c_1} \xrightarrow{g'} d_{c_2}\right).$$

where g' is as in the unique completion to the square

$$\begin{tikzcd} c \arrow[r, "g"] \arrow[d, "\eta_c"'] & c_2 \arrow[d, "\eta_{c_2}"] \\ F(d_c) \arrow[r, dashed, "G(g')"'] & G(d_{c_2}) \end{tikzcd}$$

witnessing that η_c is initial in $\mathrm{Pt}(c, G)$. This is easily seen to be a functor. The hom-set bijection

$$\mathrm{Hom}(Fc, d) \simeq \mathrm{Hom}(c, Gd)$$

is given (from left to right) by

$$f \mapsto Gf \circ \eta_c, \text{ as in } \begin{tikzcd} & G(d_c) \arrow[dd, "Gf"] \\ c \arrow[ur, "\eta_c"] \arrow[dr, "Gf\circ\eta_c"'] & \\ & G(d) \end{tikzcd}$$

and from right to left by

$$g \mapsto f_d, \text{ as in } \begin{tikzcd} & G(d_c) \arrow[dd, "G(f_d)"] \\ c \arrow[ur, "\eta_c"] \arrow[dr, "g"'] & \\ & G(d) \end{tikzcd}$$

witnessing that η_c is initial. To check naturality on the left, let $f : c_1 \to c_2$ be a map in $\mathbf{C}$. Chasing a map $g : Fc_2 \to d$ through the diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathbf{D}}(Fc_1, d) \arrow[r] & \mathrm{Hom}_{\mathbf{C}}(c_1, Gd) \\
\mathrm{Hom}_{\mathbf{D}}(Fc_2, d) \arrow[r] \arrow[u, "- \circ Ff"] & \mathrm{Hom}_{\mathbf{C}}(c_2, Gd) \arrow[u, "- \circ f"']
\end{tikzcd}$$

yields the terms

$$Gg \circ \eta_{c_2} \circ f \stackrel{?}{=} Gg \circ GFg \circ \eta_{c_1},$$

which can be seen to be equal by noting that the diagram

$$\begin{tikzcd}
 & & G(d_{c_1}) \arrow[d, "GFf"] \\
 & & G(d_{c_2}) \arrow[d, "Gg"] \\
c_1 \arrow[r, "f"'] \arrow[uurr, "\eta_{c_1}"] & c_2 \arrow[r, "Gg \circ \eta_{c_2}"'] \arrow[ur, "\eta_{c_2}"] & G(d)
\end{tikzcd}$$

commutes (the outer trapezoid on the left by definition of F; the inner triangle on the right by definition.) To see naturality on the right, let $f : d_1 \to d_2$ be map in $\mathbf{D}$. Chasing a map $g : Fc \to d_1$ through the diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathbf{D}}(Fc, d_1) \arrow[r] \arrow[d, "f \circ -"'] & \mathrm{Hom}_{\mathbf{C}}(c, Gd_1) \arrow[d, "Gf \circ -"] \\
\mathrm{Hom}_{\mathbf{D}}(Fc, d_2) \arrow[r] & \mathrm{Hom}_{\mathbf{C}}(c, Gd_2)
\end{tikzcd}$$

yields the terms

$$G(f \circ g) \circ \eta_c \stackrel{?}{=} G(f) \circ G(g) \circ \eta_c,$$

which can be seen to be equal simply by the functoriality of G. □

Definition 2.6. A functor $F : \mathbf{C} \to \mathbf{D}$ satisfies the *solution set condition* with respect to an object $d \in \mathbf{D}$ if there exists a set S_d of objects in $\mathbf{C}$ such that for all $c \in \mathbf{C}$ and $\forall f : d \to F(c)$, there is a $c' \in S_d$, a map $g : c' \to c$, and a map $f' : d \to F(c')$ such that $F(g) \circ f' = f$.

Theorem 2.7. (Freyd's general adjoint functor theorem.) *Let* $\mathbf{D}$ *be complete.* $G : \mathbf{D} \to \mathbf{C}$ *admits a left adjoint* F *if and only if* G *is continuous and satisfies the solution set condition.*

Proof. Suppose that F exists. That G is continuous is a routine Yoneda-style argument. Let $c \in \mathbf{C}$. The solution set S_c for c is just the singleton $\{d_c\} \stackrel{\text{df}}{=} \{F(c)\}$.

On the other hand, suppose that G is continuous and satisfies the solution set condition. Note that $\mathbf{D}$ being complete implies that $\mathrm{Pt}(c, G)$ is complete for each $c \in \mathbf{C}$: if a diagram $\mathcal{D} \subseteq \mathbf{D}$ becomes a diagram under some c after passing through G, then c forms a cone to $G(\mathcal{D})$ and hence admits a unique map to $G(\varprojlim \mathcal{D})$. Now, the key property of each S_c is that they are each *weakly initial families* for each $\mathrm{Pt}(c, G)$: for each $p \in \mathrm{Pt}(c, G)$, there exists some $p' \in S_c$ and a map $p' \to p$ (in $\mathrm{Pt}(c, G)$.) The product of a weakly initial family is a weakly initial object, i.e. one which admits some map, not necessarily unique, to every other object.

To complete the proof, we'll need to use the completeness of $\mathrm{Pt}(c, G)$ to obtain an initial object. Let $x = \prod S_c$, i.e. the product of the weakly initial family in $\mathrm{Pt}(c, G)$, which is a weakly initial object in $\mathrm{Pt}(c, G)$. Let $\mathrm{End}(x)$ be the diagram of all maps $x \to x$. Claim: the equalizer e as in the limit diagram

$$e \xrightarrow{i} \mathrm{End}(x),$$

is initial in $\mathrm{Pt}(c, G)$.To see this, let $p \in \mathrm{Pt}(c, G)$, and let $e \underset{g}{\overset{f}{\rightrightarrows}} p$ be two maps. Let $d \xrightarrow{j} e$ be their equalizer. There is a map $x \xrightarrow{k} d$ since x is weakly initial. By how we've set things up, the diagram

$$\begin{tikzcd} e \arrow[r, "i"] & x \arrow[r, bend left, "i \circ j \circ k"] \arrow[r, bend right, "\mathrm{id}_x"'] & x \end{tikzcd}$$

commutes. Since equalizer maps are mono, left-canceling i yields $j \circ k \circ i = \mathrm{id}_e$. Hence, since

$$e \to x \xrightarrow{i} d \xrightarrow{j} e \underset{g}{\overset{f}{\rightrightarrows}} p$$

commutes, $f = g$. Therefore, each $\mathrm{Pt}(c, G)$ has an initial object, and so a left adjoint F exists. $\square$

2.2 A definable general adjoint functor theorem

Definition 2.8. A definable functor $G : \mathbf{D} \to \mathbf{C}$ between definable categories satisfies the *definable solution set condition* if there is a uniformly definable family of sets $X_c = \varphi(\mathbb{M}, c)$ where each X_c is a weakly initial family in $\mathrm{Pt}(c, G)$.

Definition 2.9. A theory is said to have *definable Skolem functions* if for every definable set $\phi(x, y)$ there exists a definable (partial) function on the sort of y that picks out an element from the fiber $\phi(\mathbb{M}, b)$ for each b of the sort of y, if that fiber is nonempty. When additionally those functions can be made to depend only on $\phi(\mathbb{M}, b)$, i.e. if two fibers over b and b' coincide as sets then the choice function takes on the same value at b and b', the theory is said to have *definable choice functions*.

Theorem 2.10. *Let $G : \mathbf{D} \to \mathbf{C}$ be a definable functor between definable categories, with $\mathbf{D}$ definably complete. Then G admits a definable left adjoint F with the unit $\eta : 1_{\mathbf{C}} \to GF$ of the adjunction also definable if and only if G is definably continuous, satisfies the definable* *solution set condition, and there exists a definable Skolem function for the family of definable sets $\{I_c\}_{c\in\mathbf{C}}$, where I_c is the set of initial objects of* $\mathrm{Pt}(c,G)$*.*

Proof. Suppose first that $F \dashv G$ with F and the unit η definable. Then G is definably continuous. Assign to each $c \in \mathbf{C}$ the object $\left(Fc, c \xrightarrow{\eta_c} GFc\right)$ of $\mathrm{Pt}(c,G)$ which corresponds to the component of η at c.

On the other hand, suppose that G is definably continuous and satisfies the definable solution set condition. Since $\mathbf{D}$ is definably complete, and G is definably continuous, $\mathrm{Pt}(c,G)$ is definably complete for each $c \in \mathbf{C}$. The definable solution set condition ensures that each $\mathrm{Pt}(c,G)$ has an initial object. Write $\eta(c) = (d_c, x_c : c \to G(d_c))$ for the value of our definable choice function c. If $c_1 \xrightarrow{f} c_2$ is a map in $\mathbf{C}$, there is a diagram

$$\begin{tikzcd}
 & & G(d_{c_1}) \arrow[d, dashed, "G(f')"] \\
c_1 \arrow[r, "f"'] \arrow[rru, "x_{c_1}"] & c_2 \arrow[r, "x_{c_2}"'] & G(d_{c_2})
\end{tikzcd}$$

whose indicated completion uniquely witnesses that $\eta(c_1)$ is complete, and we define $F(f) = f'$, which is clearly functorial. And the choice function η can be taken verbatim to be the unit $\eta : 1_{\mathbf{C}} \to GF$. That F is actually left adjoint to G is purely formal, and so the argument from the general case may be repeated. $\square$

Remark 2.11. Note that the $\mathrm{Pt}(c,G)$ are c-definable and so form a uniformly definable family over $\mathrm{Ob}(\mathbf{C})$. Hence, the sets of initial objects of each $\mathrm{Pt}(c,G)$ are a uniformly definable subfamily.

Therefore,

Corollary 2.11.1. *If T has definable Skolem functions, then whenever $G : \mathbf{D} \to \mathbf{C}$ is a definable functor between definable categories with $\mathbf{D}$ small-complete which is right adjoint as a pure functor, its left adjoint F is definable also.*

3 Internal anafunctors in Def(T)

In the last few paragraphs it's become clear that recovering internal adjoints depends on choosing a transversal of isomorphism classes in some family of definable categories. So naively studying internal adjoints in this way requires the ambient category to satisfy some version of the axiom of choice. Anafunctors, introduced by Makkai, generalize functors to contexts where there might not be a good notion of choice. In his own words,

> Anafunctors provide solutions without introducing non-canonical choices to existence problems when data are given by universal properties. The best example

for this is the existence of an adjoint anafunctor when the “local existence criterion”[1] is satisfied.

and indeed in this section we'll find that we can prove a general adjoint functor theorem for internal anafunctors that requires no choice in the ambient category. An extremely explicit definition of anafunctors is given by Makkai in his seminal paper, but here we use a slicker definition involving spans and the regular coverage given by Bartels (2006) and Roberts (2013), which makes proving things easier. First, we introduce a notion of base change for internal categories, relative to a cover of the object-of-objects.

Definition 3.1. Let $\mathbf{C}$ be an internal category in $\mathbf{S}$ finitely complete. Let $U \xrightarrow{p} C_0$ be a map. The *base change* of $\mathbf{C}$ along p, denoted $\mathbf{C}[U]$, is given as follows:

$$\mathbf{C}[U] \overset{\text{df}}{=} \begin{cases} \mathbf{C}[U]_0 = U, \\ \mathbf{C}[U]_1 = P_s \times_{\pi_{C_1}, C_1, \pi_{C_1}} P_t, \end{cases},$$

where the latter pullback is given as in the diagram

$$\begin{tikzcd}
 & & \mathbf{C}[U]_1 \arrow[dl] \arrow[dr] & & \\
 & P_s \arrow[dl] \arrow[dr] & & P_t \arrow[dl] \arrow[dr] & \\
U \arrow[dr, "p"'] & & C_1 \arrow[dl, "s"] \arrow[dr, "t"'] & & U \arrow[dl, "p"] \\
 & C_0 & & C_0 &
\end{tikzcd}$$

where all three squares are pullbacks, with $\mathbf{C}[U]$'s source and target maps to U the upper-left and upper-right edges factoring through P_s and P_t.

Remark 3.2. In $\mathbf{Set}$, P_t and P_s are isomorphic by switching sources and targets precisely when the fibers of p are all isomorphic.

Remark 3.3. $\mathbf{C}[U]$ admits a canonical projection ρ back to $\mathbf{C}$, with ρ_0 given by $p : U \to C_0$ and ρ_1 given by the canonical map $\mathbf{C}[U]_1 \to C_1$ in the pullback diagram defining $\mathbf{C}[U]_1$ over C_1.

Definition 3.4. A *regular epimorphism* in $\mathbf{S}$ is a map $f : X \to Y$ such that f is the coequalizer of some parallel pair of maps into X.

Lemma 3.4.1. *In $\mathbf{Def}(T)$, a map is a regular epimorphism if and only if it is a definable surjection.*

Suppose first f is a definable surjection. Then

$$\begin{tikzcd} \ker(f) \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & X \arrow[r, "f"] & Y \end{tikzcd}$$

[1]As we will see, this is the anafunctor analogue of the statement that the comma categories as in the proof of the usual GAFT all have an initial object.

is a coequalizer diagram. (By using s and t we are implicitly identifying the definable equivalence relation $\ker(f)$ on X with the groupoid whose connected components are the codiscrete groupoids on $\ker(t)$-classes.) On the other hand, suppose that $f : X \to Y$ is the coequalizer of some parallel pair of maps $R \overset{s}{\underset{t}{\rightrightarrows}} X$. Since we may form images we may as well take R to be a relation $R \overset{(s,t)}{\hookrightarrow} X \times X$. That f is the coequalizer of this relation means that it is constant on R'-classes, where R' is the equivalence relation on X gotten by taking the symmetric, reflexive, and transitive closures of R while identifying points elsewhere with just themselves. This is ind-definable, and in **Set** we have a factorization

$$\begin{tikzcd}
\ker(f) \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & X \arrow[r, "f"] \arrow[dr, "\pi"'] & Y \\
 & & X/R' \arrow[u, dashed]
\end{tikzcd}$$

and since we may always form images in $\mathbf{Def}(T)$, f must be surjective.

Definition 3.5. A *regular category* $\mathbf{S}$ is one which is finitely complete, pullbacks of regular epis are regular epis, and kernel pairs admit coequalizers.

Lemma 3.5.1. $\mathbf{Def}(T)$ *is regular.*

Proof. Finite limits are constructive and are computed as in **Set**, where pullbacks of epis are epis, hence surjective. So the pullback of a definable surjection is a definable surjection, hence regular. Since images are definable, the kernel pair $\ker(f)$ of f is coequalized by the map f', which we define to be f with its codomain replaced by $\mathrm{im}(f)$. $\square$

Definition 3.6. Let $\mathbf{S}$ be a regular category. The *regular coverage* (which coincides with the canonical singleton Grothendieck pretopology on $\mathbf{C}$) is the Grothendieck pretopology whose covering families are singletons of the regular epimorphisms. The Grothendieck topology generated by the regular coverage is called the *regular topology* on $\mathbf{C}$.

Definition 3.7. Let $(\mathbf{S}, J)$ be a site such that internal categories admit base changes along covers in J. An *internal anafunctor* between internal categories $F : \mathbf{C} \to \mathbf{D}$ comprises the following data:

- **(i)** A singleton J-cover $U \to C_0$.
- **(ii)** An internal functor $\tau_F : \mathbf{C}[U] \to \mathbf{D}$.

Proposition 3.7.1. *Suppose* $\mathbf{S}$ *has pullbacks. The canonical projection* $\rho_F : \mathbf{C}[U] \to \mathbf{C}$ *is fully faithful.*

Proof. Full faithfulness is equivalent to the square

$$\begin{tikzcd}
\mathbf{C}[U]_1 \arrow[r, "(\rho_F)_1"] \arrow[d, "(s',t')"'] & C_1 \arrow[d, "(s,t)"] \\
U \times U \arrow[r, "p\times p"'] & C_0 \times C_0
\end{tikzcd}$$

being a pullback. Let $(Q, (s'', t'') : Q \to U \times U, r : Q \to C_1)$ form a cone to the above cospan. This means that we have a commutative diagram

$$\begin{tikzcd}
 & & Q \arrow[dddll, bend right, "s''"'] \arrow[ddd, bend right, "r"] \arrow[dddrr, bend left, "t''"] & & \\
 & & \mathbf{C}[U]_1 \arrow[dl] \arrow[dr] & & \\
 & P_s \arrow[dl] \arrow[dr] & & P_t \arrow[dl] \arrow[dr] & \\
U \arrow[dr, "p"'] & & C_1 \arrow[dl, "s"] \arrow[dr, "t"'] & & U \arrow[dl, "p"] \\
 & C_0 & & C_0 &
\end{tikzcd}$$

so we get unique mediating maps of cones to the cospans of the bottom two pullbacks: $Q \xrightarrow{q_s} P_s, Q \xrightarrow{q_t} P_t$. Since their compositions with π_{C_1} from P_s and P_t must equal r, this makes Q into a cone to the cospan of the top pullback as well, which yields a unique mediating map $Q \to \mathbf{C}[U]_1$. $\square$

Corollary 3.7.1. *When $\mathbf{S} = \mathbf{Def}(T)$ and J is the regular coverage, the canonical projection ρ_F is fully faithful and surjective on objects.*

Proof. $(\rho_F)_0$ is p, which is a regular epi, which is a definable surjection by the earlier lemma. $\square$

Another advantage to the approach of defining anafunctors with respect to a coverage instead of plain spans is that we obtain a very concrete description of their composition (and in the presence of canonical choices of pullbacks along covers, as in the definable setting, we get a canonical choice of composition.)

Definition 3.8. Let $F : \mathbf{C}_1 \to \mathbf{C}_2, G : \mathbf{C}_2 \to \mathbf{C}$ be anafunctors, given by

$$\begin{tikzcd}
 & \mathbf{C}_1[U_f] \arrow[dl, "\rho_F"'] \arrow[dr, "\tau_F"] & & \mathbf{C}_2[U_G] \arrow[dl, "\rho_G"'] \arrow[dr, "\tau_G"] & \\
\mathbf{C}_1 & & \mathbf{C}_2 & & \mathbf{C}_3.
\end{tikzcd}$$

Their *composite anafunctor* $GF : \mathbf{C}_1 \to \mathbf{C}_3$ will be given by

$$\begin{tikzcd}
 & & \mathbf{C}_1[U_f \times_{(\tau_F)_0,(\mathbf{C}_2)_0,p_G} U_G] \arrow[dl, "\pi_F"'] \arrow[dr, "\pi_G"] & & \\
 & \mathbf{C}_1[U_f] \arrow[dl, "\rho_F"'] \arrow[dr, "\tau_F"] & & \mathbf{C}_2[U_G] \arrow[dl, "\rho_G"'] \arrow[dr, "\tau_G"] & \\
\mathbf{C}_1 & & \mathbf{C}_2 & & \mathbf{C}_3,
\end{tikzcd}$$

where the projection functors π_F and π_G are obtained as follows: on objects, $(\pi_F)_0$ and $(\pi_G)_0$ are just the canonical projections $U_F \times_{(\mathbf{C}_2)_0} U_G \twoheadrightarrow U_F$ and $U_F \times_{(\mathbf{C}_2)_0} U_G \twoheadrightarrow U_G$. On morphisms, we induce $(\pi_F)_1$ and $(\pi_G)_1$ as in the diagrams

$$\begin{tikzcd}
 & & \mathbf{C}_1[U_F \times_{(\mathbf{C}_2)_0} U_G]_1 \arrow[dl] \arrow[d, dashed, "\overset{\mathrm{df}}{=}(\pi_F)_1"] \arrow[dr] & & \\
 & P^{U_F \times_{(\mathbf{C}_2)_0} U_G}_{s,\mathbf{C}_1} \arrow[ddl] \arrow[d, dashed] \arrow[ddr] & \mathbf{C}_1[U_F]_1 \arrow[dl] \arrow[dr] & P^{U_F \times_{(\mathbf{C}_2)_0} U_G}_{t,\mathbf{C}_1} \arrow[ddl] \arrow[d, dashed] \arrow[ddr] & \\
 & P^{U_F}_{s,\mathbf{C}_1} \arrow[d] \arrow[dr] & & P^{U_F}_{t,\mathbf{C}_1} \arrow[dl] \arrow[d] & \\
U_F \times_{(\mathbf{C}_2)_0} U_G \arrow[r, "\pi_{U_F}"] \arrow[dr] & U_F \arrow[d, "p_F"] & (\mathbf{C}_1)_1 \arrow[dl, "s"] \arrow[dr, "t"'] & U_F \arrow[d, "p_F"] & U_F \times_{(\mathbf{C}_2)_0} U_G \arrow[l, "\pi_{U_F}"'] \arrow[dl] \\
 & (\mathbf{C}_1)_0 & & (\mathbf{C}_1)_0 &
\end{tikzcd}$$

and

$$\begin{tikzcd}
 & & \mathbf{C}_1[U_F \times_{(\mathbf{C}_2)_0} U_G]_1 \arrow[dl] \arrow[d, dashed, "\overset{\mathrm{df}}{=}(\pi_G)_1"] \arrow[dr] & & \\
 & P^{U_F \times_{(\mathbf{C}_2)_0} U_G}_{s,\mathbf{C}_2} \arrow[ddl] \arrow[d, dashed] \arrow[ddr] & \mathbf{C}_2[U_G]_1 \arrow[dl] \arrow[dr] & P^{U_F \times_{(\mathbf{C}_2)_0} U_G}_{t,\mathbf{C}_2} \arrow[ddl] \arrow[d, dashed] \arrow[ddr] & \\
 & P^{U_G}_{s,\mathbf{C}_2} \arrow[d] \arrow[dr] & & P^{U_G}_{t,\mathbf{C}_2} \arrow[dl] \arrow[d] & \\
U_F \times_{(\mathbf{C}_2)_0} U_G \arrow[r, "\pi_{U_G}"] \arrow[dr] \arrow[d, "\pi_{U_F}"'] & U_G \arrow[d, "p_G"'] & (\mathbf{C}_2)_1 \arrow[dl, "s"] \arrow[dr, "t"'] & U_G \arrow[d, "p_G"] & U_F \times_{(\mathbf{C}_2)_0} U_G \arrow[l, "\pi_{U_G}"'] \arrow[dl] \arrow[d, "\pi_{U_G}"] \\
U_F \arrow[r, "(\tau_F)_0"'] & (\mathbf{C}_2)_0 & & (\mathbf{C}_2)_0 & U_F. \arrow[l, "(\tau_F)_0"]
\end{tikzcd}$$

Remark 3.9. By Lemma 2.24 of (Roberts, 2013) GF is actually a strict pullback in the 2-category of internal categories of the ambient category $\mathbf{S}$, so the composition of anafunctors is the composition of underlying spans.

Definition 3.10. The *identity anafunctor* $1_{\mathbf{C}}$ on a category $\mathbf{C}$ is given by taking the base change $\mathbf{C}[C_0]$ of $\mathbf{C}$ along $C_0 \xrightarrow{\mathrm{id}} C_0$, and setting $\tau_{\mathrm{id}_{\mathbf{C}}}$ to be the identity functor on $\mathbf{C}$. Similarly, we identify plain functors F as anafunctors by doing the same except we τ from $\mathbf{C}[C_0]$ to be F instead.

Definition 3.11. (Roberts, 2013) Given two internal anafunctors $\mathbf{C} \underset{G}{\overset{F}{\rightrightarrows}} \mathbf{D}$, an *internal natural transformation* $\eta : F \to G$, or just a *transformation*, is an internal natural transformation (which, abusing notation, we also call η) $\tau_F \circ \pi_F \xrightarrow{\eta} \tau_G \circ \pi_G$ between the two internal functors which form the left and right sides of this diagram:

$$\begin{tikzcd}
 & \mathbf{C}[U_F \times_{C_0} U_G] \arrow[dl, "\pi_F"'] \arrow[dr, "\pi_G"] & \\
\mathbf{C}[U_F] \arrow[r, "\rho_F"] \arrow[dr, "\tau_F"'] & \mathbf{C} & \mathbf{C}[U_G] \arrow[l, "\rho_G"'] \arrow[dl, "\tau_G"] \\
 & \mathbf{D}, &
\end{tikzcd}$$

where π_F and π_G are induced analogously to when they are induced when forming the composition of anafunctors.

A transformation whose component maps are all isomorphisms will be called an *isotransformation*.

3.1 Definable adjoint anafunctors

In this section we resort to using points, and so specialize our ambient category $\mathbf{S}$ to one which admits a faithful left-exact forgetful functor to $\mathbf{Set}$ (say, $\mathbf{Def}(T)$). In this setting we develop the basic theory of adjoint anafunctors, culminating in a general adjoint functor theorem.

Notation 3.12. If s is an element in U_F, we use subscripts and superscripts to denote the images of s under ρ_F and τ_F, e.g. $s_{\rho_F(s)}, s^{\tau_F(s)}, s^{\tau_F(s)}_{\rho_F(s)}$.

Definition 3.13. Let $F : \mathbf{C} \leftrightarrows \mathbf{D} : G$ be a pair of anafunctors. F is *left adjoint* to G (written $F \dashv G$) if for any s_c and v_d in U_F and U_G, we have a bijection

$$\phi_{s_c,v_d} : \mathrm{Hom}_{\mathbf{D}}(\tau_F(s_c), d) \to \mathrm{Hom}_{\mathbf{C}}(c, \tau_G(v_d))$$

which is natural in s_c and v_d in the following way: for $c \xrightarrow{h} c'$ in $\mathbf{C}$, with lifts $s_c, t_{c'}$ in U_F

and a v_d in U_G, the square

$$\begin{tikzcd}
\mathrm{Hom}_{\mathbf{D}}(\tau_F(s_c), d) \arrow[r, "\phi_{s_c,v_d}"] & \mathrm{Hom}_{\mathbf{C}}(c, \tau_G(v_d)) \\
\mathrm{Hom}_{\mathbf{D}}(\tau_F(t_{c'}), d) \arrow[r, "\phi_{t_{c'},v_d}"'] \arrow[u, "-\circ\tau_F(\rho_F^{-1}(h))"] & \mathrm{Hom}_{\mathbf{C}}(c', \tau_G(v_d)) \arrow[u, "-\circ h"']
\end{tikzcd}$$

commutes (note that specifying lifts of c and c' means that we may use the full faithfulness of ρ_F to uniquely lift h, as indicated by the notation), and for $d \xrightarrow{h} d'$ in $\mathbf{D}$ with lifts $v_d, w_{d'}$ in U_G and a s_c in U_F, the square

$$\begin{tikzcd}
\mathrm{Hom}_{\mathbf{D}}(\tau_F(s_c), d) \arrow[r, "\phi_{s_c,v_d}"] \arrow[d, "h\circ-"] & \mathrm{Hom}_{\mathbf{C}}(c, \tau_G(v_d)) \arrow[d, "\tau_G(\rho_G^{-1}(h))\circ-"] \\
\mathrm{Hom}_{\mathbf{D}}(\tau_F(s_c), d) \arrow[r, "\phi_{s_c,w_{d'}}"'] & \mathrm{Hom}_{\mathbf{C}}(c, \tau_G(w_{d'}))
\end{tikzcd}$$

commutes. Given this data, F and G are said to be *hom-set adjoint*.

Remark 3.14. Given this definition, it is natural to try to reformulate it in terms of unit and counit transformations. While (as we will see) we can recover the unit and counit from the hom-set bijections, formulating the triangle identities $F \xrightarrow{F\eta} FGF \xrightarrow{\epsilon F} F = \mathrm{id}_F$ (resp. G) in terms of transformations of anafunctors does not go through as smoothly, because it appears that there is no general way to define the precomposition of a natural transformation of anafunctors by another anafunctor.

Definition 3.15. (Composing a transformation by an anafunctor) Let $\eta : F \to G$ be a transformation of anafunctors $\mathbf{C} \to \mathbf{D}$. Let $H : \mathbf{D} \to \mathbf{E}$ be an anafunctor. Define $H\eta : HF \to HF$ as follows: form the composites HF, HF and take their pullback over $\mathbf{C}$. Note that this pullback admits a mediating map (induced by a mediating map of the pullback of covers) to $\mathbf{C}[U_F \times_{C_0} U_G]$, as indicated in the diagram

$$\begin{tikzcd}
\mathbf{C}[U_{HF} \times_{C_0} U_{HG}] \arrow[rrrr] \arrow[drr] \arrow[d, dashed] & & & & \mathbf{C}[U_G \times_{D_0} U_H] \arrow[ddll] \arrow[dd] & \\
\mathbf{C}[U_F \times_{C_0} U_G] \arrow[dr] \arrow[ddrr] & & \mathbf{C}[U_F \times_{D_0} U_H] \arrow[dl] \arrow[drr] & & & \\
& \mathbf{C}[U_F] \arrow[dl] \arrow[drr] & \mathbf{C}[U_G] \arrow[dll] \arrow[dr] & & \mathbf{D}[U_H] \arrow[dl] \arrow[dr] & \\
\mathbf{C} & & & \mathbf{D} & & \mathbf{E}.
\end{tikzcd}$$

We require η to induce a natural transformation $(U_F \times_{D_0} U_H) \times_{C_0} (U_G \times_{D_0} U_H) \to E_1$. Recall that $\mathbf{D}[U_H]_1$ is defined as the pullback

$$(D_1 \times_{s,D_0,p_H} U_H) \times_{\pi_{D_1},D_1,\pi_{D_1}} (D_1 \times_{t,D_0,p_H} U_H).$$

At the level of objects, this gives us a diagram

$$\begin{tikzcd}
& U_{HF} \times_{C_0} U_{HG} \arrow[dl] \arrow[dr, "\pi_{U_H \times U_H}"] & \\
U_F \times_{C_0} U_G \arrow[d, "\eta"'] & & U_H \times U_H \arrow[dl, "p_H{,}p_H"] \\
D_1 \arrow[r, "(s{,}t)"'] & D_0 \times D_0 &
\end{tikzcd}$$

so that taking projections to either side, as in

$$\begin{tikzcd}
& U_{HF} \times_{C_0} U_{HG} \arrow[dl] \arrow[dr, "\pi_1 \circ \pi_{U_H \times U_H}"] & \\
U_F \times_{C_0} U_G \arrow[d, "\eta"'] \arrow[dr, "(\tau_F \circ \pi_F)_0"] & & U_H \arrow[dl, "p_H"] \\
D_1 \arrow[r, "s"'] & D_0 &
\end{tikzcd}$$

and

$$\begin{tikzcd}
& U_{HF} \times_{C_0} U_{HG} \arrow[dl] \arrow[dr, "\pi_1 \circ \pi_{U_H \times U_H}"] & \\
U_F \times_{C_0} U_G \arrow[d, "\eta"'] \arrow[dr, "(\tau_G \circ \pi_G)_0"] & & U_H \arrow[dl, "p_H"] \\
D_1 \arrow[r, "t"'] & D_0 &
\end{tikzcd}$$

yields mediating maps u_s and u_t to either component of $\mathbf{D}[U_H]_1$. That these maps are fibered over D_1 follows from the fact the two cones above have the same map to D_1, and so we may form their product $(u_s \times_{D_1} u_t) : U_{HF} \times_{C_0} U_{HG} \to \mathbf{D}[U_H]_1$ over D_1. We then set

$$H\eta \stackrel{\text{df}}{=} (\tau_H)_1 \circ (u_s \times_{D_1} u_t),$$

which easily checked to be an internal natural transformation.

Remark 3.16. However, things do not go so smoothly when we take an $I : \mathbf{B} \to \mathbf{C}$ and try to form $\eta I : FI \to GI$ a transformation of anafunctors $\mathbf{B} \to \mathbf{D}$ instead. Forming compositions and taking pullbacks as before yields the diagram

$$\begin{tikzcd}
{} & {} & \mathbf{B}[(U_{FI}) \times_{B_0} (U_{GI})] \arrow[dll] \arrow[dr] & {} & {} & {} \\
\mathbf{B}[U_I \times_{C_0} U_F] \arrow[drr] \arrow[drrrr] & & & \mathbf{B}[U_I \times_{C_0} U_G] \arrow[rr] \arrow[dl] & & \mathbf{C}[U_G] \arrow[ddll, bend left] \arrow[dd] \\
& & \mathbf{B}[U_I] \arrow[dl, "p_I"'] \arrow[dr, "\tau_I"] & & \mathbf{C}[U_F] \arrow[dl, "p_F"'] \arrow[dr, "\tau_F"] & \\
& \mathbf{B} & & \mathbf{C} & & \mathbf{D}
\end{tikzcd}$$

where the only sensible thing to do, it would seem, is to find some mediating map from

$$(U_I \times_{C_0} U_F) \times_{B_0} (U_I \times_{C_0} U_G) \to U_F \times_{C_0} U_G.$$

This amounts to finding a mediating map $U_{FI} \times_{B_0} U_{GI} \to U_{FI} \times_{C_0} U_{GI}$, which generally does not exist, since there's no guarantee that the latter is also a weak pullback fibered over C_0. Note that this obstruction disappears when the cover p_I is mono, in particular if I is actually a functor. As we will see, this obstruction will also disappear when we define the triangle identities for the unit and counit of an adjunction of anafunctors.

Definition 3.17. Let $F : \mathbf{C} \leftrightarrows \mathbf{D} : \mathbf{G}$ be a pair of anafunctors. Let $\epsilon : FG \to 1_{\mathbf{D}}$ be a transformation. Define $\epsilon F : FG \circ F \to F$ via precomposition as follows:

$$\begin{tikzcd}
(U_F \times_{D_0} U_{FG}) \times_{C_0} U_F \arrow[rr] \arrow[dr, "\pi_{U_{FG}}"'] & & D_1 \\
& U_{FG} \simeq U_{FG} \times_{D_0} D_0, \arrow[ur, "\epsilon"'] &
\end{tikzcd}$$

which is easily checked to be a transformation via the naturality of ϵ.

Definition 3.18. We compose transformations of anafunctors as we do plain functors. To express this diagramatically, let F, G and H be anafunctors $\mathbf{C} \to \mathbf{D}$, and let $\eta_1 : F \to G$ and $\eta_2 : G \to H$ be transformations. The composite $\eta_2 \circ \eta_1$ is defined by pointwise composition, i.e.

$$U_F \times_{C_0} U_H \xrightarrow{\mathrm{id} \times \Delta \times \mathrm{id}} (U_F \times_{C_0} U_G) \times_{D_0} (U_G \times_{C_0} U_H) \xrightarrow{\eta_1 \times \eta_2} D_1 \times_{D_0} D_1 \xrightarrow{c} D_1.$$

Definition 3.19. Let $F : \mathbf{C} \leftrightarrows \mathbf{D} : G$ be a pair of anafunctors. We say that F and G are *unit-counit adjoint* if there exist transformations

$$1_{\mathbf{C}} \xrightarrow{\eta} GF \text{ and } FG \xrightarrow{\epsilon} 1_{\mathbf{D}}$$

such that the following equations hold:

$$\begin{tikzcd}
F \arrow[r, "F\eta"] & F \circ GF \arrow[d, "\alpha_1"] & \\
& FG \circ F \arrow[r, "\epsilon F"] & F
\end{tikzcd} \quad = \mathrm{id}_F$$

and

$$\begin{tikzcd} G \arrow[r, "\eta G"] & GF \circ G \arrow[d, "\alpha_2"] & \\ & G \circ FG \arrow[r, "G\epsilon"] & G \end{tikzcd} = \mathrm{id}_G,$$

where α_1 and α_2 are the canonical *associator isotransformations*, where, for example, the map

$$\alpha_1 : ((U_F \times_{D_0} U_G) \times_{C_0} U_F) \times_{C_0} (U_F \times_{D_0} (U_G \times_{C_0} U_F)) \to C_1$$

is induced by projecting to C_0 (along the fiber product in the middle) and composing by the identity map $C_0 \to C_1$.

Theorem 3.20. *Let $F : \mathbf{C} \leftrightarrows \mathbf{D} : G$ be a pair of anafunctors. Then F and G are hom-set adjoint if and only if they are unit-counit adjoint.*

Proof. ($\Rightarrow$) From the hom-set bijections, obtain $\eta : U_{1_\mathbf{C}} \times_{C_0} (U_F \times_{D_0} U_G) \to C_1$ by

$$(s^d_c, v_d) \xmapsto{\eta} \phi(d \xrightarrow{\mathrm{id}_d} d) \overset{\mathrm{df}}{=} c \xrightarrow{\overline{\mathrm{id}_d}} \tau_G(v_d)$$

and $\epsilon : (U_G \times_{C_0} U_F) \times_{D_0} D_0 \to D_1$ by

$$(v^c_d, s_c) \xmapsto{\epsilon} \phi^{-1}(c \xrightarrow{\mathrm{id}_c} c) \overset{\mathrm{df}}{=} \tau_F(s_c) \xrightarrow{\overline{\mathrm{id}_c}} d.$$

For the triangle identity at F, let $c \in C_0$ and $d \in D_0$ and take lifts (s^d_c, v_d) and $s'_{\tau_G(v_d)}$. Then applying $F\eta$ yields

$$\eta(s^d_c, v_d) \longmapsto \tau_F\left(\rho_F^{-1}\eta(s^d_c, v_d)\right) =\!=\!= \left(\tau_F(s^d_c) \xrightarrow{\tau_F(\rho_F^{-1}\overline{\mathrm{id}_d})} \tau_F(s'_{\tau_G(v_d)})\right).$$

Since v_d and $s'_{\tau_F(v_d)}$ are also fibered over C_0, applying ϵF yields

$$\tau_F\left(s'_{\tau_G(v_d)}\right) \xrightarrow{\overline{\mathrm{id}_{\tau_G(v_d)}}} d,$$

and we can see that

$$d \xrightarrow{\tau_F\left(\rho_F^{-1}\overline{id_d}\right)} \tau_F\left(s'_{\tau_G(v_d)}\right) \xrightarrow{\overline{\mathrm{id}_{\tau_G(v_d)}}} d = \mathrm{id}_d$$

by forming the naturality square at s_c, v^c and $v_{\tau_F(s_c)}$

$$\begin{tikzcd} \mathrm{Hom}_\mathbf{C}(c, c) \arrow[r] \arrow[d, "\overline{\mathrm{id}_{\tau_F(s_c)}} \circ -"'] & \mathrm{Hom}_\mathbf{D}(\tau_F(s_c), \rho_G(v^c)) \arrow[d, "\tau_F\rho_F^{-1}\overline{\mathrm{id}_{\tau_F(s_c)}} \circ -"] \\ \mathrm{Hom}_\mathbf{C}(c, \tau_G(v_{\tau_F(s_c)}))) \arrow[r] & \mathrm{Hom}_\mathbf{D}(\tau_F(s_c), \tau_F(s_c)) \end{tikzcd}$$

and chasing id_c.

Similarly, taking lifts (v^c, s_c) and $v'_{\tau_F(s_c)}$ and applying ηF yields

$$\eta(s_c, v'_{\tau_F(s_c)}) = c \xrightarrow{\overline{\mathrm{id}_{\tau_F(s_c)}}} \tau_G(v'_{\tau_F(s_c)}),$$

and applying $G\epsilon$ yields

$$\tau_G(v'_{\tau_F(s_c)}) \xrightarrow{\tau_G(\rho_G^{-1}(\overline{\mathrm{id}_c}))} \tau_G(v^c) = c,$$

and to see that

$$\tau_G(\rho_G^{-1}(\overline{\mathrm{id}_c})) \circ \overline{\mathrm{id}_{\tau_F(s_c)}} = \mathrm{id}_c,$$

$$\begin{tikzcd}
\mathrm{Hom}_{\mathbf{C}}(c, \tau_G(v'_{\tau_F(s_c)})) \arrow[r] \arrow[d, "\tau_G(\rho_G^{-1}\overline{\mathrm{id}_c})\circ -"'] & \mathrm{Hom}_{\mathbf{D}}(\tau_F(s_c), \tau_F(s_c)) \arrow[d, "\overline{id_c}\circ -"] \\
\mathrm{Hom}_{\mathbf{C}}(c, c) \arrow[r] & \mathrm{Hom}_{\mathbf{D}}(\tau_F(s_c), \tau_F(s_c)),
\end{tikzcd}$$

where chasing $\overline{\mathrm{id}_{\tau_F(s_c)}}$ yields

$$\phi\left(\tau_G(\rho_G^{-1}\overline{\mathrm{id}_c}) \circ \overline{\mathrm{id}_{\tau_F(s_c)}}\right) = \overline{\mathrm{id}_c},$$

so that $\tau_G(\rho_G^{-1}\overline{\mathrm{id}_c}) \circ \overline{\mathrm{id}_{\tau_F(s_c)}} = \mathrm{id}_c$, as required.

($\Rightarrow$) On the other hand, given η and ϵ, define ϕ and ϕ^{-1} as follows: we know that ϕ^{-1} would have to satisfy, for any $c_1 \xrightarrow{f} c_2$ with lifts s_{c_1}, s'_{c_2}, and $v_d^{c_2}$,

$$\begin{tikzcd}
\mathrm{Hom}_{\mathbf{C}}(c_1, c_2) \arrow[r] & \mathrm{Hom}_{\mathbf{D}}(\tau_F(s_{c_1}), d) \\
\mathrm{Hom}_{\mathbf{C}}(c_2, c_2) \arrow[r] \arrow[u, "-\circ f"] & \mathrm{Hom}_{\mathbf{D}}(\tau_F\rho_F^{-1}(f), d) \arrow[u, "-\circ\tau_F(\rho_F^{-1}(f))"']
\end{tikzcd}$$

so that in particular $\phi^{-1}(f) = \overline{\mathrm{id}_{c_2}} \circ \tau_F(\rho_F^{-1}(f))$. Similarly, for any $g : d_1 \to d_2$, with lifts v_{d_1}, v'_{d_2}, and $s_c^{d_1}$,

$$\begin{tikzcd}
\mathrm{Hom}_{\mathbf{D}}(d_1, d_1) \arrow[r] \arrow[d, "g\circ -"'] & \mathrm{Hom}_{\mathbf{C}}(c, \tau_G(v_{d_1})) \arrow[d, "\tau_G\rho_G^{-1}g\circ -"] \\
\mathrm{Hom}_{\mathbf{D}}(d_1, d_2) \arrow[r] & \mathrm{Hom}_{\mathbf{C}}(c, \tau_G(v'_{d_2})),
\end{tikzcd}$$

so that in particular, $\phi(g) = \tau_G\rho_G^{-1}\overline{\mathrm{id}_{d_1}}$. Now, $\phi \circ \phi^{-1}(f) \stackrel{?}{=} f$ becomes

$$\tau_G\rho_G^{-1}\left(\tau_F(s_c) \xrightarrow{\overline{\mathrm{id}_{c_2}}\circ\tau_F\rho_F^{-1}f} d\right) \circ \overline{\mathrm{id}_{\tau_F(s_c)}} \stackrel{?}{=} f,$$

which is precisely the triangle identity at F. Similarly, $\phi^{-1} \circ \phi(g) \stackrel{?}{=} g$ becomes

$$\overline{\mathrm{id}_{\tau_G(v'_{d_2})}} \circ \tau_F \rho_F^{-1} \left(c \xrightarrow{\tau_G \rho_G^{-1}(g) \circ \overline{\mathrm{id}_{d_1}}} \tau_G(v'_{d_2}) \right) \stackrel{?}{=} g,$$

which is precisely the triangle identity at G. It is easy to verify that ϕ is natural, so we conclude the proof. $\square$

Definition 3.21. (Categories of points for anafunctors). Let $G : \mathbf{D} \to \mathbf{C}$ be an anafunctor. Let c be an object of $\mathbf{C}$. The *category of c-points* $\mathrm{Pt}(c, G)$ of G is given by:

$$\begin{cases} \text{Objects: pairs } (v_d, c \xrightarrow{p} \tau_G(v_d)). \\ \text{Morphisms: } \mathrm{Hom}((v_d, p), (v'_{d'}, p')) \text{ consists of those maps } d \xrightarrow{f} d' \text{ such that } \tau_F(\rho_G^{-1}(f)) \circ p = p'. \end{cases}$$

That the categories of points have initial objects is precisely what Makkai calls the "local existence of a left adjoint." The following is the natural analogue of our earlier lemma on plain adjoint functors, and provides a converse to 2.1 of Makkai.

Theorem 3.22. *Let $G : \mathbf{D} \to \mathbf{C}$ be an anafunctor. G admits left adjoint F if and only if for all $c \in \mathbf{C}$, $\mathrm{Pt}(c, G)$ has an initial object.*

Proof. ($\Rightarrow$) The unit of the adjunction at c

$$(c, s_c, v_{\tau_F(s_c)}) \mapsto \left(c \xrightarrow{\eta(c)} \tau_G(v_{\tau_F(s_c)}) \right)$$

is initial in $\mathrm{Pt}(c, G)$:

$$\begin{tikzcd}
 & \tau_G(v_{\tau_F(s_c)}) \arrow[dd, dashed] & & & \tau_F(s_c) \arrow[dd] \\
c \arrow[ur, "\eta(c)"] \arrow[dr, "p"'] & & \Longleftrightarrow & \tau_F(s_c) \arrow[ur, "\mathrm{id}_{\tau_F(s_c)}"] \arrow[dr, "\phi^{-1}(p)"'] & \\
 & \tau_G(v_d) & & & d
\end{tikzcd}$$

where the indicated completion is uniquely determined by taking tranposes across ϕ.

($\Leftarrow$) Construct the anafunctor F as follows: let U_F be the coproduct of the objects-of-initial objects from each $\mathrm{Pt}(c, G)$. The projection $p_F : U_F \to C_0$ is gotten by just forgetting everything but the c from a c-point. On $s_c = c \xrightarrow{p_c} \tau_G(v'_{d'})$, $\tau_F(s_c)$ is just d'. On a morphism $s_c \xrightarrow{f} s'_{c'}$, $\tau_F(f)$ is just $\rho_G(g)$, where g uniquely completes the diagram

$$\begin{tikzcd}
c \arrow[r, "p_c"] \arrow[dr, "\rho_F(f)"'] & \tau_G(v'_{d'}) \arrow[dr, dashed, "\tau_G(g)"] & \\
 & c' \arrow[r, "p_{c'}"] & \tau_G(v''_{d''})
\end{tikzcd}$$

witnessing that p_c is initial.

Now we give the hom-set bijections. Fix $s_c = c \xrightarrow{p_c} \tau_G(v'_{d'})$ and v_d. Let $g : d' \to d$ be a map. Define $\phi(g)$ as the composite

$$\begin{tikzcd}
\tau_G(v'_{d'}) \arrow[r, "\tau_G(\rho_G^{-1}(g))"] & \tau_G(v_d) \\
c \arrow[u, "p_c"] \arrow[ur, dashed] &
\end{tikzcd}$$

and similarly $\phi^{-1}(c \xrightarrow{p'} \tau_G(v_d))$ as g, as in the composite

$$\begin{tikzcd}
c \arrow[r, "p'"] \arrow[dr, "p_c"'] & \tau_G(v_d) \\
& \tau_G(v'_{d'}). \arrow[u, dashed, "\tau_G \rho_G^{-1}(g)"']
\end{tikzcd}$$

These maps are well-defined because s_c is initial; for the same reason, they also invert each other.

Now naturality: for $f : c \to c'$, and lifts s_c, s'_c, v_d, chasing $g : \tau_F(s'_{c'}) \to d$ around the square

$$\begin{tikzcd}
\mathrm{Hom}_{\mathbf{D}}(\tau_F(s_c), d) \arrow[r, "\phi"] & \mathrm{Hom}_{\mathbf{C}}(c, \tau_G(v_d)) \\
\mathrm{Hom}_{\mathbf{D}}(\tau_F(s'_{c'}), d) \arrow[r, "\phi"'] \arrow[u, "- \circ \tau_F(\rho^{-1}(f))"] & \mathrm{Hom}_{\mathbf{C}}(c', \tau_G(v_d)) \arrow[u, "- \circ f"']
\end{tikzcd}$$

yields the tentative equality

$$\tau_G \rho_G^{-1}\left(g \circ \tau_F \rho_F^{-1}(f)\right) \circ p_c \stackrel{?}{=} \tau_G \rho_G^{-1}(g) \circ p_{c'} \circ f,$$

which is seen to be true because they are an initial map from p_c in $\mathrm{Pt}(c, G)$. Similarly, taking $g : d \to d'$, lifts $s_c, v_d, v'_{d'}$ and chasing an $f : \tau_F(s_c) \to d$ through the square

$$\begin{tikzcd}
\mathrm{Hom}_{\mathbf{D}}(\tau_F(s_c), d) \arrow[r, "\phi^{-1}"] \arrow[d, "g \circ -"'] & \mathrm{Hom}_{\mathbf{C}}(c, \tau_G(v_d)) \arrow[d, "\tau_G \rho_G^{-1}(g) \circ -"] \\
\mathrm{Hom}_{\mathbf{D}}(\tau_F(s_c), d') \arrow[r, "\phi^{-1}"'] & \mathrm{Hom}_{\mathbf{C}}(c, \tau_G(v'_{d'}))
\end{tikzcd}$$

yields the tentative equality

$$\tau_G \rho_G^{-1}(g \circ f) \circ p_{c'} \stackrel{?}{=} \left(\tau_G \rho_G^{-1}(g) \circ \tau_G \rho_G^{-1}(f)\right) \circ p_c$$

which is again seen to be true because they are an initial map from $p_{c'}$ in $\mathrm{Pt}(c', G)$. $\square$

Remark 3.23. When, as in **Set** and **Def**(T), we have a canonical choice of coproduct to construct U_F (projecting with an existential in the latter; just taking a union in the former), the left adjoint F is canonical, and does not depend on choice.

Before stating and proving the general adjoint functor theorem, we have to define what it means for an anafunctor to preserve limits.

Definition 3.24. Let $G : \mathbf{D} \to \mathbf{C}$ be an anafunctor. Let $J : \mathbf{J} \to \mathbf{D}$ be a diagram of shape $\mathbf{J}$ in $\mathbf{D}$. Suppose the limit $\varprojlim J$ exists in $\mathbf{D}$. G is said to *preserve the limit* $\varprojlim J$ if for every lift L of J along ρ_G, as in

$$\begin{tikzcd} & \mathbf{D}[U_G] \arrow[d, "\rho_G"] \\ \mathbf{J} \arrow[ur, dashed, "L"] \arrow[r, "J"'] & \mathbf{D}, \end{tikzcd}$$

the limit of $\tau_G \circ L$ exists in $\mathbf{C}$ and for any $v_{\varprojlim J} \in U_G$ lifting a limit to J in $\mathbf{D}$,

$$\tau_G(v_{\varprojlim J}) \simeq \varprojlim \tau_G \circ L.$$

G is said to be small-continuous (resp. for κ in place of small for any infinite cardinal κ) if it preserves all limits of small (resp. κ-sized) diagrams. If it preserves all limits of internal functors from internal categories, it is said to be *internally continuous.*

Proposition 3.24.1. *Suppose that F and G form an adjoint pair of anafunctors $\mathbf{C} \leftrightarrows \mathbf{D}$. Then G is internally continuous.*

Proof. Let $J : \mathbf{J} \to \mathbf{D}$, $v_{\varprojlim J}$ be as above; let L be a lift of J along ρ_G, and let c and s_c be any object in $\mathbf{C}$ and a lift of it in U_F. Then:

$$\begin{aligned} \mathrm{Hom}_{\mathbf{C}}(c, \tau_G v_{\varprojlim J}) &\simeq \mathrm{Hom}_{\mathbf{D}}(\tau_F s_c, \varprojlim J) \\ &\simeq \varprojlim \left(\mathrm{Hom}_{\mathbf{D}}(\tau_F s_c, -) : \mathbf{J} \to \mathbf{S}\right) \\ &\simeq \varprojlim \left(\mathrm{Hom}_{\mathbf{C}}(c, \tau_G \circ L(-)) : \mathbf{J} \to \mathbf{S}\right) \\ &\simeq \mathrm{Hom}_{\mathbf{D}}(c, \varprojlim(\tau_G \circ L)), \end{aligned}$$

and so by the Yoneda lemma,

$$\tau_G v_{\varprojlim J} \simeq \varprojlim \left(\tau_G \circ L\right).$$

□

Theorem 3.25. *Let $\mathbf{G} : \mathbf{D} \to \mathbf{C}$ be an anafunctor on $\mathbf{D}$ an internally complete category. Then $\mathbf{G}$ admits a left adjoint F if and only if G is internally continuous and for each $c \in C_0$, $\mathrm{Pt}(c, G)$ has a weakly initial family of objects.*

Proof. The proof proceeds as in the case of plain functors, and with all the ingredients we have so far, we only need to show that with our assumptions, the internal continuity of G ensures the internal completeness of the categories of points $\mathrm{Pt}(c, G)$.

But a diagram $J' : \mathbf{J} \to \mathrm{Pt}(c, G)$ is just a diagram of c-points where the transition maps lie in the image of the functor, i.e. consists of triangles of the form

$$\begin{tikzcd}
& \tau_G(v_{j_i}) \arrow[dd, "\tau_G(v_{j_i} \to v_{j_k})"] \\
c \arrow[ur] \arrow[dr] & \\
& \tau_G(c_{j_k})
\end{tikzcd}$$

The v_{j_i}'s and the transition maps between them form the data of a functor $\mathbf{J} \xrightarrow{L} \mathbf{D}[U_G]$. Composing by ρ_G yields a diagram $J = \rho_G \circ L : \mathbf{J} \to \mathbf{D}$, so L lifts J, and continuity of G gives

$$\tau_G v_{\varprojlim J} \simeq \varprojlim \tau_G \circ L.$$

Since the diagram J' in $\mathrm{Pt}(c, G)$ already makes c into a cone to $\tau_G \circ L$, it admits a unique mediating map to $c \xrightarrow{p} \varprojlim \tau_G \circ L \simeq \tau_G v_{\varprojlim J}$ for any $v_{\varprojlim J}$ lifting the limit of J in $\mathbf{D}$ along ρ_G. Since any other cone to J' in $\mathrm{Pt}(c, G)$ may be forgotten into a cone to $\tau_G \circ L$ in $\mathbf{C}$, J' has limit p in $\mathrm{Pt}(c, G)$. $\square$

Corollary 3.25.1. *In particular, if a definable functor G satisfies the hypotheses of the theorem, its canonical adjoint anafunctor is definable as well, regardless of the presence of definable Skolem functions.*

3.2 Morita equivalences

In this section we continue assuming that the ambient category $\mathbf{S}$ is concrete in the sense that it admits a faithful left-exact forgetful functor to **Set**.

Notation 3.26. If $f : c \to c'$ is a map in $\mathbf{C}$ and we have an anafunctor $F : \mathbf{C} \to \mathbf{D}$, then specifying lifts $s_c, s'_{c'}$ of c and c' lets us (by the full faithfulness of ρ_F) uniquely lift f to a map $s_c \to s'_{c'}$. Previously we've suppressed the extra data of s_c and $s'_{c'}$ and have denoted the lift of f as $\rho_F^{-1}(f)$; now we write it as

$$\left(f \nearrow s_c, s'_{c'}\right) : s_c \to s'_{c'}.$$

Definition 3.27. An anafunctor $F : \mathbf{C} \to \mathbf{D}$ is a *Morita equivalence* if both ρ_F and τ_F are full, faithful and surjective on objects.

Definition 3.28. An anafunctor $F : \mathbf{C} \to \mathbf{D}$ is an *anaequivalence* if τ is full, faithful, and surjective on objects.

Remark 3.29. By considerations of Makkai in his paper, this notion of anaequivalence is equivalent to having a pseudo-inverse anafunctor with isotransformations from the composites to the identity anafunctors.

Definition 3.30. An anafunctor $F : \mathbf{C} \to \mathbf{D}$ is *presaturated* if for each $c \in C_0$ and each $s_c \in U_F$ lifting c along ρ_F and each isomorphism $\phi : \tau_F(s_c) \to d$ in $\mathbf{D}$, there is some ${s'}_c^d$ such that

$$\phi = \tau_f \left(\mathrm{id}_c \nearrow s_c, {s'}_c^d \right).$$

Definition 3.31. An anafunctor $F : \mathbf{C} \to \mathbf{D}$ is *saturated* if it is presaturated and additionally the ${s'}_c^d$ as in the previous definition is unique.

Definition 3.32. Let $F : \mathbf{C} \to \mathbf{D}$ be an anafunctor. The *presaturation* $F^{\#'}$ of F is gotten by follows: we define

$$U_{F\#'} \overset{\mathrm{df}}{=} \coprod_{c \in C_0} \left\{ (s_c^d, \phi) \,\middle|\, \phi : d \xrightarrow{\sim} d' \right\} \simeq \coprod_{d \in D_0} \coprod_{s_c^d \in \tau_F^{-1} d} \left(d \middle/ \mathrm{core}(\mathbf{D}) \right)_0 \simeq U_F \times_{(\tau_F)_0, Y_0, s} \mathrm{core}(\mathbf{D})_1,$$

and given an $f : c_1 \to c_2$ and its lift

$$f : c_1 \to c_2 \nearrow (s_{c_1}^{d_1}, \phi_1 : d_1 \to d_1'), (s_{c_2}^{d_2}, \phi_2 : d_2 \to d_2'),$$

we define $\tau_{F\#'} : \mathbf{C}[U_{F\#'}] \to \mathbf{D}$ by

$$\left(f \nearrow (s_{c_1}^{d_1}, \phi_1 : d_1 \to d_1'), (s_{c_2}^{d_2}, \phi_2 : d_2 \to d_2') \right) \mapsto \left(d_1' \xrightarrow{\phi_1^{-1}} d_1 \xrightarrow{f \nearrow s_{c_1}^{d_1}, s_{c_2}^{d_2}} d_2 \xrightarrow{\phi_2} d_2' \right).$$

This is functorial: in case the lift of f is the identity, ϕ_1 and ϕ_2 coincide; if we form the composition with a lifted g between $(s_{c_2}^{d_2}, \phi_2)$ and $(s_{c_3}^{c_3}, \phi_3)$, then ϕ_2 and ϕ_2^{-1} cancel in the middle.

Lemma 3.32.1. *The presaturation $F^{\#'}$ of F is presaturated.*

Proof. Let $c \in C_0$ and $(s_c^d, \phi : d \to d')$ lift c in $U_{F\#'}$. Let $\phi' : d' \to d''$ be an isomorphism. We require some $\left(s_c^{d'''}, \phi'' : d''' \to d'' \right)$ such that

$$d' \xrightarrow{\phi^{-1}} d \xrightarrow{\mathrm{id}_c \nearrow s_c^d, s_c^{d'''}} d''' \xrightarrow{\phi''} d'' = d' \xrightarrow{\phi'} d''.$$

We can take $\left(s_c^d, \phi' \circ \phi : d \to d'' \right)$. ∎

Definition 3.33. To obtain the *saturation* $F^{\#}$ from the presaturation $F^{\#'}$, we quotient $U_{F\#'}$ by the relation

$$(s_c^{d_1}, \phi_1) \sim ({s'}_c^{d_2}, \phi_2) \iff t(\phi_1) = t(\phi_2) = d_3 \text{ and the diagram } \begin{tikzcd} d_1 \arrow[rr, "\left(\mathrm{id}_c \nearrow s_c^{d_1}, {s'}_c^{d_2}\right)"] \arrow[dr, "\phi_1"'] & & d_2 \arrow[dl, "\phi_2"] \\ & d_3 & \end{tikzcd} \text{ commutes,}$$

and we define the functor $\tau_{F\#} : \mathbf{C}[U_{F\#}] \to \mathbf{D}$ the same as we did for the presaturation.

Lemma 3.33.1. *If τ_F is full and faithful, then so is $\tau_{F\#'}$, and if τ_F is essentially surjective then $\tau_{F\#'}$ is surjective on objects.*

Proof. Presaturation clearly extends the image of a functor to include everything isomorphic to anything already in its image, which is the second item. To see the first, just use that either precomposing or composing by an isomorphism is an isomorphism between contra- and covariant hom-functors, so in particular induces bijections on hom-sets. Doing both at the same time amounts to composing one of these bijections after the other, which gives full faithfulness. □

Corollary 3.33.1. *Two internal categories are Morita equivalent if and only if they are anaequivalent if and only if they are presaturated anaequivalent.*

Corollary 3.33.2. *In addition, if we are working in* $\mathbf{Def}(T)$*, then if* T *eliminates imaginaries the three conditions above are all also equivalent to being saturated anaequivalent.*

4 Internal diagrams in $\mathbf{Def}(T)$

When doing ordinary (small) category theory, i.e. category theory internal to **Set**, there are useful functors from those small—internal—categories, to a large—external—one, notably hom-functors. For example, if we specialize to a group **G** internal to **Set**, the contravariant hom-functor is precisely **G**'s right action ontiself; similarly, if **G** is a groupoid instead, its right action on itself can be construed as a **Set**-valued functor on **G**.

An *internal diagram* generalizes this notion to **S**-valued functors on categories internal to **S**.

Definition 4.1. Let **S** be finitely complete, and let **C** be an internal category of **S**. An *internal diagram* $P : \mathbf{C} \to \mathbf{S}$ comprises the following data:

(i) An object (abusing notation) P of **S**, equipped with a map (the *anchor map*) $P \xrightarrow{p_0} C_0$ (interpretation: P is a C_0-indexed collection of objects, i.e. p_0 gives the object part of the functor $\mathbf{C} \to \mathbf{S}$), and

(ii) an *action map* $C_1 \times_{s,C_0,p_0} P \xrightarrow{p_1} P$ (interpretation: this describes the morphism part of the functor $\mathbf{C} \to \mathbf{S}$),

subject to the following conditions:

(i) The diagram

$$\begin{tikzcd} C_1 \times_{s,C_0,p_0} P \arrow[r, "p_1"] \arrow[d] & P \arrow[d, "p_0"] \\ C_1 \arrow[r, "t"'] & C_0 \end{tikzcd}$$

commutes. (Interpretation: the image of a point $p \in P$ fibered over $c \in C_0$ under a map $c \to c'$ will be fibered over c'.)

(ii) The diagram

$$\begin{tikzcd}
 & C_1 \times_{s,C_0,p_0} P \arrow[dr, "p_1"] & \\
P \arrow[ur, "{(i,\mathrm{id}_P)}"] \arrow[rr, "\mathrm{id}_P"'] & & P
\end{tikzcd}$$

commutes. (Interpretation: identities act as identities (so P the functor preserves them.)

(iii) The diagram

$$\begin{tikzcd}
(C_1 \times_{s,C_0,t} C_1)_{s\pi_2,C_0,p_0} P \arrow[r, "(\mathrm{id}_{C_1} \times p_1)"] \arrow[d, "(c \times \mathrm{id}_P)"'] & C_1 \times_{s,C_0,p_0} P \arrow[ddl, "p_1"] \\
C_1 \times_{s,C_0,p_0} P \arrow[d, "p_1"'] & \\
P &
\end{tikzcd}$$

commutes. (Interpretation: the functor P, i.e. the action map p_1, is compatible with composition.)

As remarked by Johnstone in his section on internal category theory in *Topos Theory*, internal diagrams naturally carry the data of their internal categories of points, so we abuse notation and identify the internal diagram $\mathbf{P}$ with its category of points $\mathbf{P}$ equipped with an internal forgetul functor $\mathbf{P} \xrightarrow{p} \mathbf{C}$. (So, for example, the action groupoid of a definable group action is always definable.) In particular, since every internal category has a canonical internal diagram on itself induced by its action on itself, there is a canonical $\mathbf{C}$-torsor for every internal category $\mathbf{C}$. In particular, this $\mathbf{C}$-torsor is always definable. What is not always certain, however (and this is where Barr-exactness, i.e. elimination of imaginaries comes in) is whether or not there is an "object-of-isomorphism-classes", i.e. if we can form the quotient $C_0/_{\simeq}$.

[NOTE TO SELF: THIS SECTION NEEDS TO BE REWORKED.]

Definition 4.2. (Internal category of points of an internal diagram) Let F be an internal diagram $\mathbf{C} \to \mathbf{S}$. We can naturally obtain an internal category $\mathbf{F}$ from F as follows: the object-of-objects is F_0, the object-of-morphisms is $F_0 \times_{\gamma_0,C_0,s} C_1$, the source and target maps for $\mathbf{F}$ are the projection as in the above pullback and the action map from the data of the internal diagram. Setting $\gamma_1 \overset{\mathrm{df}}{=} \pi_{C_1}$ as in the above pullback makes (γ_0, γ_1) into a canonical projection functor from $\mathbf{F} \to \mathbf{C}$, and in case $\mathbf{S} = \mathbf{Set}$, $\mathbf{F}$ is seen to be the canonical category of elements of F and γ the canonical projection functor.

Given an internal diagram on an internal category (think: group actions; for example G-flows are precisely internal diagrams on internal groups in **Top** which factor through the inclusion of **ComHausTop**) we get an analogue of the orbit-stabilizer theorem—in fact, a direct generalization of it—in the same way that we can write ordinary presheaves as colimits of representables, so that the study of internal diagrams on $\mathbf{C}$ reduces to the study of the irreducible internal diagrams on $\mathbf{C}$. These will be the $\mathbf{C}$-*torsors*.

Proposition 4.2.1. (Internal orbit-stabilizer) *[see johnstone]*

Proof. $\square$

Definition 4.3. If we can form images in the ambient category $\mathbf{S}$, we say that an internal diagram $\mathbf{P} : \mathbf{C} \to \mathbf{S}$ is *transitive* if the canonical map

$$\Gamma(p_1) \to \operatorname{im}(\pi_{P \times P})$$

is a cover, where $\Gamma(p_1)$ is the graph of the action map p_1 (gotten as the obvious pullback) and $\pi_{P\times P}$ is as in the following pullback square:

$$\begin{tikzcd}
C_1 \times_{C_0 \times C_0} P \times P \arrow[r] \arrow[d, "\pi_{P\times P}"'] & C_1 \arrow[d, "(s,t)"] \\
P \times P \arrow[r, "p_0 \times p_0"] & C_0 \times C_0.
\end{tikzcd}$$

In more familiar language, this means "for every arrow $f : C \to C'$, for every p, p' belonging to $p_0^{-1}(C), p_0^{-1}(C')$, there exists some $f' : C \to C'$ such that $p_1(f', p) = p'$."

Proposition 4.3.1. *Suppose* $\mathbf{S}$ *has images. Then* $\mathbf{P} : \mathbf{C} \to \mathbf{S}$ *is irreducible if and only if it is transitive.*

Proof. $\square$

Remark 4.4. To translate this into the definable setting, replace "internal" with "definable", "diagram" with "action", and "exact completion" with "T^{eq}".

4.1 Categories of definable diagrams and generalized imaginary sorts

Now we specialize to $\mathbf{S} = \mathbf{Def}(T)$.

Definition 4.5. Analogously to the notation $\mathbf{G}$-$\mathbf{Set}$ for the category of $\mathbf{G}$-sets, i.e. of internal diagrams on a group $\mathbf{G}$ viewed as a groupoid internal to $\mathbf{Set}$, we write $\mathbf{C}$-$\mathbf{Def}(T)$ for the category of definable diagrams on a definable category $\mathbf{C}$.

Recall that a *Morita morphism* between categories is a fully faithful functor which is surjective on objects.

Definition 4.6. Let $F : \mathbf{X} \to \mathbf{Y}$ be a Morita morphism. Define the functor

$$(-) \searrow_F : \mathbf{X}\text{-}\mathbf{Def}(T) \to \mathbf{Y}\text{-}\mathbf{Def}(T)$$

("extension along F") by

$$\left(P \xrightarrow{f} P'\right) \searrow_F \stackrel{\text{df}}{=} (P/\sim) \xrightarrow{f/\sim} (P'/\sim),$$

where

(i) $p \sim p' \iff F(p_0(p)) = F(p_0(p')) \stackrel{\text{df}}{=} y$ and there exists a $\sigma : p_0(p) \to p_0(p')$ such that $\sigma = \left(\mathrm{id}_y \nearrow_{p_0(p), p_0(p')}\right)$ and $p_1(\sigma, p) = p'$; and

(ii) we define $f/\sim$ by identifying f with its graph, i.e. as the canonical pullback $P \times_{f, P', \mathrm{id}_{P'}} P$, which is naturally a definable diagram on $\mathbf{X}$ when equipped with the product action: $g.(p, f(p)) \stackrel{\text{df}}{=} (g.p, g.f(p) = f(g(p)))$.

Lemma 4.6.1. *$(-) \searrow_F$, as described, is actually a functor.*

Proof. $P/\sim$ (resp. P') is naturally equipped with the structure of a definable diagram on $\mathbf{Y}$ with

anchor map $p_0/\sim([p]) \stackrel{\text{df}}{=} Fp_0(p)$ (where we have written $[p]$ for the $\sim$-class of p), and

action map $p_1/\sim(\sigma, [p]) \stackrel{\text{df}}{=}$ (for $\sigma : p_0/\sim([p]) \to y'$)

$$\left[p_1\left(\left(\sigma \nearrow_{p_0(p), x}\right), p\right)\right]_\sim$$

where x is a lift of y' along F. To see that the choice of x does not matter, take another x' which lifts y' and compute:

$$\left[p_1\left(\left(\sigma \nearrow_{p_0(p), x}\right), p\right)\right]_\sim = \left[p_1\left(\left(\sigma \nearrow_{p_0(p), x'}\right), p\right)\right]$$

$$\iff p_1\left(\left(\mathrm{id}_{y'} \nearrow_{x, x'}\right) \circ \left(\sigma \nearrow_{p_0(p), x}\right), p\right) = p_1\left(\left(\sigma \nearrow_{p_0(p), x'}\right), p\right),$$

which follows from the functoriality of lifts. Since we've already identified $f/\sim$ with its graph, this also shows that $f/\sim$ is a morphism of definable diagrams over $\mathbf{Y}$.

Furthemore, it's clear that $\mathrm{id}_P \nearrow_F$ is again the identity, and that F is compatible with composition.

We verify that this data satisfies the definition of an internal diagram on $\mathbf{Y}$:

(i)

$$\begin{tikzcd}
(g : Fp_0(p) \to y', [p]) \arrow[r, mapsto] \arrow[dd, mapsto] & \left[p_1\left(g \nearrow_{p_0(p), x'}, p\right)\right] \arrow[d, mapsto] \\
 & Fx' \arrow[d, equal] \\
g \arrow[r, mapsto] & y'
\end{tikzcd}$$

commutes,

(ii)

$$\begin{tikzcd}
 & \left(\mathrm{id}_{Fp_0(p)}, [p]\right) \arrow[r, mapsto] & \left[p_1\left(\mathrm{id}_{Fp_0(p)} \nearrow Fp_0(p), x', p\right)\right] \arrow[d, equal] \\
{[p]} \arrow[ur, mapsto] \arrow[rr, mapsto] & & {[p]}
\end{tikzcd}$$

commutes, and

(iii)

$$\begin{tikzcd}
\left((f : y' \to y'', g : Fp_0(p) \to y'), [p]\right) \arrow[r, mapsto] \arrow[d, mapsto] & \left(f, \left[p_1\left(g \nearrow p_0(p), x', p\right)\right]\right) \arrow[dd, mapsto] \\
\left(f \circ g, [p]\right) \arrow[d, mapsto] & \\
\left[p_1\left(f \circ g \nearrow p_0(p), x'', [p]\right)\right] \arrow[r, equal] & \left[p_1\left(f \nearrow x', x''', p_1\left(g \nearrow p_0(p), x', p\right)\right)\right]
\end{tikzcd}$$

commutes, where x'' and x''' are both arbitrary points in the preimage of y'' under F_0.

□

Definition 4.7. Let $F : \mathbf{X} \to \mathbf{Y}$ be a Morita morphism. Define the functor

$$(-) \nearrow F : \mathbf{Y}\text{-}\mathbf{Def}(T) \to \mathbf{X}\text{-}\mathbf{Def}(T)$$

("lift along F") by

$$\left(P \xrightarrow{f} P'\right) \nearrow F \overset{\mathrm{df}}{=} \left(P \times_{p_0, Y_0, F_0} X_0\right) \xrightarrow{f \times_{Y_0} X_0} \left(P' \times_{p_0', Y_0, F_0} X_0\right),$$

where as before we have identified f with its graph.

Lemma 4.7.1. *$(-) \nearrow F$, as described, is actually a functor.*

Proof. $P \times_{p_0, Y_0, F_0} X_0$ is naturally equipped with the structure of a definable diagram on $\mathbf{X}$: the anchor map is just the projection $\pi_{X_0} : P \times_{p_0, Y_0, F_0} X_0 \twoheadrightarrow X_0$, and the action map $X_1 \times_{s, X_0, \pi_{X_0}} (P \times_{p_0, Y_0, F_0} X_0) \to P \times_{p_0, Y_0, F_0} X_0$ is given by

$$(g, (p, x)) \mapsto (p_1(F_1(g), p), t(g)),$$

and it is also easy to see that $\mathrm{id}_P \nearrow F = \mathrm{id}_{P \times_{Y_0} X_0}$ and that $(-) \nearrow F$ is compatible with composition.

We verify that this data satisfies the definition of an internal diagram on $\mathbf{X}$:

(i)

$$\begin{tikzcd}
\big(g : x \to x', (p, x)\big) \arrow[r, mapsto] \arrow[dd, mapsto] & \big(p_1(F(g), p), x'\big) \arrow[d, mapsto] \\
& x' \arrow[d, equal] \\
g \arrow[r, mapsto] & x'
\end{tikzcd}$$

commutes,

(ii)

$$\begin{tikzcd}
& (\mathrm{id}_x, (p, x)) \arrow[r, mapsto] & (p, x) \arrow[d, equal] \\
(p, x) \arrow[ur, mapsto] \arrow[rr, mapsto] & & (p, x)
\end{tikzcd}$$

commutes, and

(iii)

$$\begin{tikzcd}
\big((f : x' \to x'', g : x \to x'), (p, x)\big) \arrow[rr, mapsto] \arrow[d, mapsto] & & \big(f, (p_1(F(g), p), x')\big) \arrow[d, mapsto] \\
\big(f \circ g, (p, x)\big) \arrow[r, mapsto] & \big(p_1(F(f \circ g), p), x''\big) \arrow[r, equal] & \big(p_1(F(f), p_1(F(g), p)), x''\big)
\end{tikzcd}$$

commutes.

□

Proposition 4.7.1. *Let* $\mathbf{X} \xrightarrow{F} \mathbf{Y}$ *be a definable functor. If* F *is a Morita morphism, then* F *induces an equivalence of categories*

$$(-) \searrow_F : \mathbf{X}\text{-}\mathbf{Def}(T) \simeq \mathbf{Y}\text{-}\mathbf{Def}(T) : (-) \nearrow_F .$$

Proof. Let P be a definable diagram on $\mathbf{X}$. For each $y \in Y_0$, extending P along F just collapses the fibers $\left(p_0^{-1}(x)\right)_{x \in F_0^{-1}(y)}$ (which are all isomorphic) to a single fiber; lifting back along F takes the product of this single fiber with $(x)_{x \in F_0^{-1}(y)}$. Let P' name the definable diagram on $\mathbf{X}$ that results from this process. The isomorphism $\eta_P : P \xrightarrow{\sim} P'$ is defined by:

$$\Gamma(\eta_P)\left(p, ([p'], x)\right) \iff [p] = [p'] \text{ and } p_0(p) = x.$$

We check that $(\eta_P)_{P \in \mathbf{X}\text{-}\mathbf{Def}(T)}$ is natural in P: to see

$$\begin{tikzcd} P_1 \arrow[r, "\eta_{P_1}"] \arrow[d, "f"'] & P_1' \arrow[d, "f'"] \\ P_2 \arrow[r, "\eta_{P_2}"'] & P_2' \end{tikzcd}$$

commutes, chase a $p \in P_1$, viz.

$$\begin{tikzcd} p \arrow[r, mapsto] \arrow[dd, mapsto] & ([p], p_0(p)) \arrow[d, mapsto, "{[f] \times_{Y_0} X_0}"] \\ & ([f(p)], p_0(p)) \arrow[d, equal] \\ f(p) \arrow[r, mapsto] & ([f(p)], p_0(f(p))) \end{tikzcd} ,$$

where the last equality is due to the equivariance of f (i.e. its naturality as a transformation of functors $P_1 \to P_2$.)

On the other hand, if we start with P as a definable diagram on $\mathbf{Y}$, lifting P along F replaces each fiber $p_0^{-1}(y)$ with $F_0^{-1}(y)$-many copies of itself. If $x \to x'$ lifts id_Y, then $p_1 \nearrow F(x \to x', (p,x)) = (p, x')$, hence extending back along F just collapses the fibers again. Let P' name the result of this process. Define the isomorphism $\epsilon_P : P' \to P$ as follows:

$$\Gamma(\epsilon_P)(p', p) \iff \text{for all } (q,x) \text{ which project to } p', \text{ q = p.}$$

We check that $(\epsilon_P)_{P \in \mathbf{Y}\text{-}\mathbf{Def}(T)}$ is natural in P: to see

$$\begin{tikzcd} P_1' \arrow[r, "\epsilon_{P_1}"] \arrow[d, "f'"'] & P_1 \arrow[d, "f"] \\ P_2' \arrow[r, "\epsilon_{P_2}"'] & P_2 \end{tikzcd}$$

commutes, chase a $q \in P_1'$, viz.

$$\begin{tikzcd} {[(p,x)]} \arrow[r, mapsto] \arrow[d, mapsto] & p \arrow[d, mapsto] \\ {[(f(p),x)]} \arrow[r, mapsto] & f(p). \end{tikzcd}$$

□

Corollary 4.7.1. *Let T interpret a saturated anaequivalence of categories between $\mathbf{X}$ and $\mathbf{Y}$. Then there is an equivalence of categories*

$$i_T : \mathbf{X}\text{-}\mathbf{Def}(T) \simeq \mathbf{Y}\text{-}\mathbf{Def}(T) : j_T.$$

Proof. Let the saturated anaequivalence be given by

$$\begin{tikzcd} & \mathbf{X}[U_F] \arrow[dl, "\rho_F"'] \arrow[dr, "\tau_F"] & \\ \mathbf{X} & & \mathbf{Y}. \end{tikzcd}$$

By the previous proposition, we have equivalences

$$\mathbf{X}\text{-}\mathbf{Def}(T) \simeq \mathbf{X}[U_F]\text{-}\mathbf{Def}(T) \simeq \mathbf{Y}\text{-}\mathbf{Def}(T).$$

□

Theorem 4.8. *Suppose T interprets a saturated anaequivalence $\mathbf{X} \leftarrow \mathbf{X}[U_F] \rightarrow \mathbf{Y}$. Let T_P and T_Q be expansions of T by an extra sort P and Q, such that the expansion includes function symbols (p_0, p_1), (q_0, q_1) and sentences which give P and Q the structures of definable diagrams on $\mathbf{X}$ and $\mathbf{Y}$, respectively. Then if there are interpretations J_{PQ} and J_{QP}*

$$\begin{tikzcd}
T_P \arrow[rr, shift left, "J_{PQ}"] & & T_Q \arrow[ll, shift left, "J_{QP}"] \\
& T \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

of T_P and T_Q in each other over T such that $J_{PQ}(P, p_0, p_1) = j_{T_Q}(Q, q_0, q_1)$ and $J_{QP}(Q, q_0, q_1) = i_{T_P}(P, p_0, p_1)$, then in fact they form a bi-interpretation $J_{PQ} : T_P \simeq T_Q : J_{QP}$ of T_P and T_Q over T.

Proof.

$$P \mapsto j_{T_Q}(Q) = \left(Q \nearrow \tau_F\right) \searrow \rho_F = \left(Q \times_{q_0.Y_0.(\tau_F)_0} (\mathbf{X}[U_F])_0\right) \Big/ \sim_{\rho_F}$$

$$\mapsto (i_{T_P}(P) \times_{X_0} X_0) \Big/ \sim_{\rho_F} = j_{T_P} \circ i_{T_P}(P),$$

where we have used that interpretations are logical functors, hence commute with finite limits and taking images (in this case, under the quotient map $(-) \Big/ \sim_{\rho_F}$); the argument that Q is definably isomorphic to $J_{PQ} \circ J_{QP}(Q)$ is entirely analogous.

We induce the unit $\eta : 1_{\mathbf{Def}(T_P)} \rightarrow J_{QP} \circ J_{PQ}$ by setting the component $\eta_{x=_S x}$ of η at any sort S of T to be the identity, and the component η_P of η at P to be $P \xrightarrow{\sim} j_{T_P} \circ i_{T_P}(P)$; for a tuple of sorts in T_P we just take the corresponding tuple of (components of) η. For a definable set $K \in \mathbf{Def}(T_P)$ of sorts $\overline{s}$, we induce η_K by restricting $\eta_{\prod_{s \in \overline{s}} x =_s x}$, i.e. by precomposing this by the canonical identification of K inside $\prod_{s \in \overline{s}} x =_s x$, and then taking its image.

This is natural precisely when, for any $K_1 \xrightarrow{f} K_2$,

$$\forall k_1, k_2 \in K_1, K_2, \Gamma(f)\big(k_1, k_2\big) \iff (J_{QP} \circ J_{PQ}\Gamma(f))\left(\eta_{K_1}(k_1), \eta_{K_2}(k_2)\right).$$

Since we have at least two constants and therefore definable characteristic maps for definable sets, this is equivalent to:

$$\forall x \left(K(x) \iff (J_{QP} \circ J_{PQ}K)(\eta_K(x))\right),$$

for all definable sets $K \in \mathbf{Def}(T_P)$. Since η is the identity on anything from a sort of T, this is equivalent to:

$$\forall \overline{x} \in T \; \forall \overline{p} \in P \left(K(\overline{x}, \overline{p}) \leftrightarrow (J_{QP} \circ J_{PQ}K)(\overline{x}, \eta_P(\overline{p}))\right). \tag{1}$$

Since η_P is an isomorphism of definable diagrams on $\mathbf{X}$, this already holds for the graphs of p_0 and p_1, i.e.

$$\left(\models p_0(p) = x \leftrightarrow (j_{T_P} \circ i_{T_P} p_0)(\eta_P(p)) = x\right), \text{ and}$$

$$\Big(\models p_1(g,p) = p' \leftrightarrow (j_{T_P} \circ i_{T_P} p_1)\,(g, \eta_P(p)) = \eta_P(p') \Big).$$

We can see, using an induction on the complexity of formulas, that this is enough: these graphs (and equality in P, which is preserved by η_P) are the only new atomic relations, so (1) holds whenever K is atomic; the class of formulas satisfying (1) is easily seen to be closed under negation and conjunction, and to see that this class is closed under taking existentials, let $K(\overline{x}, \overline{p})$ satisfy (1), and consider

$$\overset{?}{\models} (\forall \overline{x} \backslash x, \forall \overline{p} \backslash p) \Big[\exists (x,p) K(\overline{x}, \overline{p}) \leftrightarrow \exists (x, \eta_P(p))\,(J_{QP} \circ J_{PQ} K)\,(\overline{x}, \eta_P(\overline{p}))\Big].$$

If $(\widetilde{x}, \widetilde{p}) \in \exists(x,p)K(\overline{x}, \overline{p})$, then there must exist (x,p) such that $(\widetilde{x}x, \widetilde{p}p) \in K(\overline{x}, \overline{p})$, which means $(\widetilde{x}x, \eta_P(\widetilde{p})\eta_P(p)) \in (J_{QP} \circ J_{PQ} K)\,(\overline{x}, \eta_P(\overline{p}))$, hence $(\widetilde{x}, \widetilde{p} \in (\exists x, \exists \eta_P(p))\,(J_{QP} \circ J_{PQ} K)\,(\overline{x}, \overline{p})$.

As before, the argument that the counit ϵ is also natural is entirely analogous, so we have a bi-interpretation $J_{PQ} : T_P \simeq T_Q : J_{QP}$ over T, as required. □

Corollary 4.8.1. *Suppose that* $\mathbf{X}$ *and* $\mathbf{Y}$ *as above are groupoids. Let* T_P *and* T_Q *be the expansions of* T *by the generalized imaginary sorts associated to* $\mathbf{X}$ *and* $\mathbf{Y}$*, so that* P *and* Q *are groupoid torsors of* $\mathbf{X}$ *and* $\mathbf{Y}$*, respectively. Then* T_P *and* T_Q *are bi-interpretable over* T*.*

Proof. $i_{T_P}(P)$ is a torsor of $\mathbf{Y}$ and $j_{T_Q}(Q)$ is a torsor of $\mathbf{X}$. We should like interpretations $J_{PQ} : T_P \to T_Q$ over T and $J_{QP} : T_Q \to T_P$ via $P \mapsto j_{T_Q}(Q)$ and $Q \mapsto i_{T_P}(P)$. We check that these maps preserve sentences and are hence actually interpretations.

For any two models M_P and M_Q of T_P and T_Q extending a model M_T of T, $P(M_P) \simeq j_{T_Q}(M_Q)$ as internal diagrams over the internal category $\mathbf{X}(M_T)$ in $\mathbf{Set}$. Let η_P name this isomorphism. Extend η_P to a map $\eta : M_P \to M_Q$ over M_T by making it the identity on M_T.

We argue as when we showed the naturality of the unit in the proof of the preceding theorem. We want to say that J_{PQ} and η satisfy

$$M_P \models K(\overline{y}) \iff M_Q \models J_{PQ}(K)(\eta(\overline{y}))$$

for all formulas K in the language of T_P and tuples of points $\overline{y}$ from M_P; we can separate variables according to whether their sort is P or from T, and rewrite the above as

$$M_P \models K(\overline{x}, \overline{p}) \iff M_Q \models J_{PQ}(K)(\overline{x}, \eta_P(\overline{p})).$$

Since η_P is an isomorphism of internal diagrams on $\mathbf{X}(M_T)$, this is satisfied when we take K to be equality in P or the graph relation of p_0 or p_1. Since these are the only new atomic relations, the above holds whenever K is atomic. The class of formulas for which the above holds is clearly closed under negation and conjunction. If the above holds for $K(\overline{x}, \overline{p})$ for $\overline{x}$ and $\overline{p}$ tuples of points, then

$$M_P \models (\exists x, p) \big[K(\widetilde{x}, \widetilde{p} \big]$$

(where x and p are appropriately-sorted variables and $\widetilde{x}$ and $\widetilde{p}$ are the appropriate truncations of $\overline{x}$ and $\overline{p}$)

$$\iff \text{ there exist points (abusing notation) } x, p \text{ such that } M_P \models K(\widetilde{x}x, \widetilde{p}p)$$

$$\iff M_Q \models J_{PQ}(K)\,(\widetilde{x}x, \eta_P(\widetilde{p})\eta_P(p)) \iff M_Q \models (\exists x, q)\,[J_{PQ}(K)(\widetilde{x}, \eta_P(\widetilde{p})]\,,$$

where q is a variable of the same sort as $\eta_P(p)$ (the point), i.e. is precisely $J_{PQ}(p)$ (the variable.) In particular this works when $\overline{x} = x$ and $\overline{p} = p$, so J_{PQ} induces interpretations of models $M_P \to M_Q$ for every pair of models M_P and M_Q extending a model M_T of T. In particular, J_{PQ} preserves those sentences which are true in all models of M_P, and hence is an interpretation $T_P \to T_Q$ over T; we can argue analogously that J_{QP} is an interpretation $T_Q \to T_P$ over T, and hence by the theorem, T_P and T_Q are bi-interpretable over T. □

5 The axiom of choice in $\mathbf{Def}(T)$ and equivalences of internal categories

Definition 5.1. An *equivalence of categories* $\mathbf{C} \simeq \mathbf{D}$ is the data

$$\left(F : \mathbf{C} \to \mathbf{D}, G : \mathbf{D} \to \mathbf{C}, \eta : 1_{\mathbf{C}} \xrightarrow{\sim} GF, \epsilon : FG \xrightarrow{\sim} 1_{\mathbf{D}}\right).$$

Given this data, F and G are said to be *pseudo-inverse* to each other, and either is said to form or be part of an equivalence of categories. η is called the unit and ϵ the counit. The equivalence is said to be definable if all the data are definable.

With the axiom of choice, a functor $F : \mathbf{C} \to \mathbf{D}$ is part of an equivalence of categories if and only if it is full, faithful, and essentially surjective. It turns out that the internalization of this statement to $\mathbf{Def}(T)$ is true as well (when all epimorphisms are definable surjections, otherwise with the regular coverage.)

Theorem 5.2. *Every definable surjection in T admits a definable section if and only if every definable functor $F : \mathbf{C} \to \mathbf{D}$ between definable categories $\mathbf{C}, \mathbf{D}$ in T which is full, faithful, and essentially surjective admits a definable pseudoinverse $G : \mathbf{D} \to \mathbf{C}$ which forms with F a definable equivalence of categories $\mathbf{C} \simeq \mathbf{D}$.*

Proof. ($\Leftarrow$). Let $X_0 \xrightarrow{f} Y_0$ be a definable surjection. Consider the following categories:

$$\mathbf{X} \overset{\text{df}}{=} \begin{cases} X_0 = X_0 \\ X_1 = X_0 \times_{f, Y_0, f} X_0 \\ s = \pi_1, t = \pi_2 \text{ (for the above pullback)} \\ c\left((x_1, x_2), (x_2, x_3)\right) = (x_1, x_3) \end{cases}$$

and

$$\mathbf{Y} \overset{\text{df}}{=} \begin{cases} Y_0 = Y_0 \\ Y_1 = Y_0 \\ s = \mathrm{id}_{Y_0}, t = \mathrm{id}_{Y_0} \\ c = \mathrm{id}_{Y_0} \end{cases}$$

and the functor

$$F : \mathbf{X} \to \mathbf{Y} \overset{\mathrm{df}}{=} \begin{cases} F_0 = f : X_0 \to Y_0 \\ F_1 = f \circ s. \end{cases}$$

Given a pseudoinverse $G : \mathbf{Y} \to \mathbf{X}$, we require that

$$Y_0 \xrightarrow{G_0} X_0 \xrightarrow{F_0} Y_0 = \mathrm{id}_{Y_0} .$$

We have a counit isomorphism $\epsilon : FG \simeq 1_{\mathbf{Y}}$, so the diagram

$$\begin{tikzcd} Y_1 \times_{Y_1} Y_1 \arrow[r, "c"] & Y_1 \\ Y_1 \arrow[u, "(F_0 \circ G_0, \epsilon \circ t)"] \arrow[r, "(\epsilon \circ s, \mathrm{id}_{Y_1})"'] & Y_1 \times_{Y_0} Y_1 \arrow[u, "c"'] \end{tikzcd}$$

commutes. By definition, this diagram can be rewritten as

$$\begin{tikzcd} Y_0 \arrow[r, "\mathrm{id}_{Y_0}"] & Y_0 \\ Y_0 \arrow[u, "(\epsilon \circ t, F_0 \circ G_0)"] \arrow[r, "(\mathrm{id}_{Y_0}, \epsilon \circ s,)"'] & Y_0 \arrow[u, "\mathrm{id}_{Y_0}"'] \end{tikzcd}$$

and since s and t are the identity morphisms, identifying components gives that $\mathrm{id}_{Y_0} = F_0 \circ G_0$, as required.

($\Rightarrow$). If $\mathbf{C}$ is a definable category, then its core

$$\mathrm{core}(\mathbf{C}) \overset{\mathrm{df}}{=} \begin{cases} \mathrm{core}(\mathbf{C})_0 = C_0, \\ \mathrm{core}(\mathbf{C})_1 = \{ f \in C_1 \mid f \text{ an isomorphism} \} \end{cases}$$

is definable also, with the source and target maps $\mathrm{core}(s)$ and $\mathrm{core}(t)$ just the restrictions of s and t from $\mathbf{C}$. A fully faithful essentially surjective definable functor $F : \mathbf{C} \to \mathbf{D}$ must satisfy:

(i) For each $c_1, c_2 \in C_0$, the induced definable map

$$\mathrm{Hom}_{\mathbf{C}}(c_1, c_2) \to \mathrm{Hom}_{\mathbf{D}}(F(c_1), F(c_2))$$

is a bijection, and

(ii) F preserves and reflects isomorphisms, and so restricts to a fully faithful functor between cores.

(iii) The map $\mathrm{core}(t) \circ \pi_{\mathrm{core}(\mathbf{D})_1}$ as in the following diagram involving the pullback square

$$\begin{tikzcd} P \arrow[r, "\pi_{\mathrm{core}(\mathbf{D})_1}"] \arrow[d, "\pi_{\mathrm{core}(\mathbf{C})_0}"'] & \mathrm{core}(\mathbf{D})_1 \arrow[r, "\mathrm{core}(t)"] \arrow[d, "\mathrm{core}(s)"] & \mathrm{core}(\mathbf{D})_0 \\ \mathrm{core}(\mathbf{C})_0 \arrow[r, "F_0"'] & \mathrm{core}(\mathbf{D})_0 & \end{tikzcd}$$

is a definable surjection.

To construct G, proceed as follows:

Step 1. Take a definable section s_1 to $\mathrm{core}(t) \circ \pi_{\mathrm{core}(\mathbf{D})_1}$. This yields, for each object $d \in D_0$, a $c_d \in C_0$ and an isomorphism $F(c_d) \xrightarrow{\sigma_d} d$.

Step 2. Define G as follows:

$$G\left(d_1 \xrightarrow{f} d_2\right) \stackrel{\mathrm{df}}{=} \left(c_{d_1} \xrightarrow{F^{-1}\left(\sigma_{d_2}^{-1} \circ f \circ \sigma_{d_1}\right)} c_{d_2} \right).$$

To check functoriality, note that G preserves identities, and let $d_1 \xrightarrow{f} d_2 \xrightarrow{g} d_3$. Let c_d name $s_1 \circ \pi_{\simeq}(d)$, and let σ_d name $s_2(d)$. Taking isomorphisms to the $F(c_{d_i})$, we get the diagram

$$\begin{tikzcd} d_1 \arrow[r, "f"] & d_2 \arrow[r, "g"] & d_3 \\ F(c_{d_1}) \arrow[u, "\sigma_{d_1}"] & F(c_{d_2}) \arrow[u, "\sigma_{d_2}"] & F(c_{d_3}) \arrow[u, "\sigma_{d_3}"] \end{tikzcd}$$

and so we see that by the functoriality of F^{-1} (whence full faithfulness) that G is functorial. From full faithfulness we also get that F reflects isomorphisms, so that there is always an isomorphism $c \simeq GFc$, in fact $F^{-1}\left(\sigma_{F(c)}^{-1}\right)$. This gives a definable function $C_0 \to C_1$. To see this is natural transformation, let $c_1 \xrightarrow{f} c_2$ be a map in $\mathbf{C}$; then form the square

$$\begin{tikzcd} c_1 \arrow[r, "F^{-1}\left(\sigma_{F(c_1)}^{-1}\right)"] \arrow[d, "f"'] & GF(c_1) \arrow[d, "GFf"] \\ c_2 \arrow[r, "F^{-1}\left(\sigma_{F(c_2)}^{-1}\right)"'] & GF(c_2) \end{tikzcd}$$

and note that

$$GFf\left(F^{-1}\sigma_{F(c_1)}^{-1}\right) = F^{-1}\left(\sigma_{F(c_2)}^{-1} \circ f \circ \sigma_{F(c_1)}\right) \circ F^{-1}\left(\sigma_{F(c_1)}^{-1}\right) = F^{-1}\left(\sigma_{F(c_2)}^{-1} \circ f\right),$$

so the diagram commutes. On the other hand, if $d \in \mathbf{D}$, then $FG(d) = F(c_d)$, and we already have a family of isomorphisms $\{\sigma_d : F(c_d) \to d\}$, and so a definable function $D_0 \to D_1$. To see that this is a natural transformation, let $d_1 \xrightarrow{f} d_2$ be a map in $\mathbf{D}$, and note that by definition of G,

$$FGf = F(c_{d_1}) \xrightarrow{\sigma_{d_1}} d_1 \xrightarrow{f} d_2 \xrightarrow{\sigma_{d_2}^{-1}} F(c_{d_2}),$$

so that the diagram

$$\begin{tikzcd}
F(c_{d_1}) \arrow[r, "\sigma_{d_1}"] \arrow[d, "\sigma_{d_1}"'] & d_1 \arrow[ddd, "f"] \\
d_1 \arrow[d, "f"'] & \\
d_2 \arrow[d, "\sigma_{d_2}^{-1}"'] & \\
F(c_{d_2}) \arrow[r, "\sigma_{d_2}"'] & d_2
\end{tikzcd}$$

evidently commutes. This completes the proof. □

Remark 5.3. A pseudoinverse G constructed in this way is also always right adjoint to F.

Remark 5.4. The first part of the above proof actually works for any finitely complete category $\mathbf{C}$. Carrying out the second part in this generality is much harder. While we can characterize full faithfulness as a pullback, our characterization of "essentially surjective" doesn't translate over as easily: we need an internal notion of core, which may not exist (though it always does in the definable setting.) If $\mathbf{C}$ has "elimination of imaginaries" i.e. is Barr-exact, then I think this obstruction disappears.

Proposition 5.4.1. *When epimorphisms in* $\mathbf{Def}(T)$ *are precisely the definable surjections,* T *has definable Skolem functions if and only if* T *satisfies the external axiom of choice.*

Proof. That having definable Skolem functions implies the external axiom of choice is clear. In the other direction, let $\phi(x, y)$ be a definable set in T, so that there is a canonical map

$$\phi(\mathbb{M}_x, \mathbb{M}_y) \xrightarrow{\pi_Y} \mathbb{M}_y,$$

which yields a (partial) section $y \overset{s}{\mapsto} (f(y), y)$, so that

$$\pi_X \circ s : \mathbb{M}_y \to \mathbb{M}_y$$

gives a definable Skolem function for $\phi(x, y)$. □

Corollary 5.4.1. *By the above proof,* T *admits definable Skolem functions if and only if every definable surjection admits a definable section.*

Proposition 5.4.2. *If a theory* T *defines two constants* c_1 *and* c_2*, then epimorphisms in* $\mathbf{Def}(T)$ *are precisely the definable surjections.*

Proof. If f is not a surjection, consider the definable set $Y' \underset{\text{def}}{\subseteq} Y$ defined by

$$Y' \overset{\text{df}}{=} \{y \in Y \,\big|\, \nexists x \text{ s.t. } f(x) = y\},$$

and the functions

$$g_1 : Y \to \{c_1, c_2\} \text{ by } \begin{cases} y \mapsto c_1 \text{ if } y \in Y', \\ y \mapsto c_2 \text{ if } y \in Y \backslash Y' \end{cases}$$

and

$$g_2 : Y \to \{c_1, c_2\} \text{ by } y \mapsto c_2.$$

Since g_1 and g_2 are definable, $g_1 \circ f = g_2 \circ f$ and $g_1 \neq g_2$. □

Definition 5.5. Let $\mathbf{C}$ be a category. A *skeleton* of $\mathbf{C}$ is a full subcategory $\mathbf{C}'$ of $\mathbf{C}$ such that C_0' meets each $\simeq$-class of C_0 exactly once.

Question 5.6. It's known that the syntactic category of a first-order theory is a Heyting category, and so base-change functors have right adjoints. When all these right adjoints preserve epimorphisms, the ambient category satisfies the *internal axiom of choice*: all objects are internally projective. What does this mean in model-theoretic terms?

6 A remark on notions of groupoid torsor

The notion of a category torsor we have used gives, when specialized to groupoids, a somewhat different notion of a groupoid torsor than the one used elsewhere. Here I explain the relation between the two, and show that they coincide in the case that seems to be the only one that shows up in practice.

Definition 6.1. A category is *connected* if for every two objects X, Y in the category, there is a map $X \to Y$.

Definition 6.2. Let $\mathbf{G}$ be a groupoid and let $\mathbf{C}$ be a category with terminal object $\mathbf{1}$. A $\mathbf{G}$-*torsor in* $\mathbf{C}$ is a faithful functor $\mathbf{G} \to \mathbf{C}$ such that for each connected component $\mathbf{G}^0$ of $\mathbf{G}$, $\mathrm{Pt}\left(\mathbf{1}_{\mathbf{C}}, F \restriction \mathbf{G}^0\right)$ is connected.

Definition 6.3. Let $\mathbf{G}$ be a small groupoid and let Y be a set. A $\mathbf{G}$-torsor *over* Y comprises the data

$$\left(\begin{array}{c} P \\ \Big\downarrow{\scriptstyle \pi} \\ Y \end{array}, a : P \longrightarrow G_0, (- \cdot -) : G_1 \times_{s, G_0, a} P \longrightarrow P \right)$$

where P is a set over Y, a is called the *anchor map*, and $(- \cdot -)$ is called the *action map* with $(g \cdot -) : P \to P$ an automorphism of P over Y for each $g \in G_1$. Furthermore, we must have that:

For all $p_1, p_2 \in P$ such that $\pi(p_1) = \pi(p_2)$, there exists a unique $g \in G_1$ such that $g \cdot p_1 = p_2$.

When $\mathbf{G}$ is small, there is a natural way to turn $\mathbf{G}$-torsors F in $\mathbf{Set}/Y$ into $\mathbf{G}$-torsors over Y: take

$P \overset{\text{df}}{=} \bigsqcup_{x \in G_0} F(x) = \{(p, x)$ with x naming which $F(x)$ p comes from$\}, \pi : P \to Y$ the disjoint union of the pr

$$a((p, x)) \overset{\text{df}}{=} x,$$

and

$$g \cdot (p, x) \overset{\text{df}}{=} \begin{cases} F(g)(p) \text{ if } x = \mathrm{dom}(g) \\ \text{identity otherwise.} \end{cases}$$

Remark 6.4. However, neither of these notions of $\mathbf{G}$-torsor subsumes the other.

Example 6.5. Let $\mathbf{G}$ be a groupoid with more than one connected component. Consider the $\mathbf{G}$-set $P \to G_0$ where P is the disjoint union of all automorphism groups of G and the action between fibers are just equivariant bijections of H-sets, where H is the automorphism group of the relevant connected component. This is a $\mathbf{G}$-torsor in $\mathbf{Set}/\mathbf{1}$. However, any two points in different connected components will still be fibered over the same element, so this does not give a $\mathbf{G}$-torsor over $\mathbf{1}$.

Example 6.6. Let $\mathbf{G} = \mathbf{B}\mathbb{Z}$ (i.e. the groupoid with one object which carries $\mathbb{Z}$ as its automorphism group), and consider the action $\mathbb{Z} \curvearrowright \mathbb{Z} \sqcup \mathbb{Z}$, i.e. the coproduct of the action of $\mathbb{Z}$ on itself with itself in $\mathbb{Z}$-$\mathbf{Set}$. This is a $\mathbf{B}\mathbb{Z}$-torsor over $\mathbf{2}$, but not a $\mathbf{B}\mathbb{Z}$-torsor in $\mathbf{Set}/\mathbf{2}$: pick any two sections $\mathbf{2} \to \mathbb{Z} \sqcup \mathbb{Z}$ such that the two points of one section are a different distance apart than the two points of the other.

Proposition 6.6.1. *Let $G_0/\simeq$ denote the connected components of $\mathbf{G}$, and suppose $\mathbf{G}$ is small. Then a $\mathbf{G}$-torsor in $\mathbf{Set}$ is equivalent to a $\mathbf{G}$-torsor over $G_0/\simeq$ with π canonical.*

Proof. Let F be a $\mathbf{G}$-torsor in $\mathbf{Set}$. Take

$P \overset{\text{df}}{=} \bigsqcup_{x \in G_0} F(x) = \{(p, x)$ with x naming the $F(x)$ where p comes from$\}, \pi : P \to G_0/\simeq$ by $\pi((p, x)) \overset{\text{df}}{=} [x]_{\simeq}$

$$a((p, x)) \overset{\text{df}}{=} x,$$

and

$$g \cdot (p, x) \overset{\text{df}}{=} \begin{cases} F(g)(p) \text{ if } x = \mathrm{dom}(g), \\ \text{identity otherwise.} \end{cases}$$

To satisfy the last requirement for being a $\mathbf{G}$-torsor over Y, fix p_1 and p_2 belonging to $F(x_1)$ and $F(x_2)$ for some $x_1, x_2 \in G_0$ in the same connected component. Then there are sections $s_1 : Y \to F(x_1)$ and $s_2 : Y \to F(x_2)$ where everything points at p_1 and p_2, so by definition there is some $g : x_1 \to x_2$ such that $F(g) \circ s_1 = s_2$, hence $F(g)(p_1) = p_2$. Since F is faithful, the restriction of F to the subcategory on x_1 is a transitive group action, so for any $x_1 \xrightarrow{f} x_2$,

$F(f)$ is either fixed-point-free or the identity. Hence, for any $g' : x_1 \to x_2$, g' and g must either disagree at all points or coincide everywhere. Hence, our g is unique.

On the other hand, given the data of a $\mathbf{G}$-torsor over $G_0/\simeq$ with the canonical projection π, build F as follows: if x_1 and x_2 both lie in the same connected component c, then define

$$F\left(x_1 \xrightarrow{g} x_2\right) \overset{\text{df}}{=} \left(\begin{tikzcd} a^{-1}(x_2) \arrow[rr, "p \mapsto g \cdot p"] \arrow[dr] & & a^{-1}(x_2) \arrow[dl] \\ & \{c\} & \end{tikzcd} \right),$$

which is easily checked to be functorial. To see that F is transitive, note that if $F(g_1) = F(g_2)$, then they must agree at some point, and hence $g_1 = g_2$ by the uniqueness clause of the second notion of $\mathbf{G}$-torsors. Now let s_1 and s_2 be sections $\mathbf{1} \simeq \{c\} \xrightarrow{s_1} F(x_1)$ and $\mathbf{1} \simeq \{c\} \xrightarrow{s_2} F(x_2)$, for x_1 and x_2 both in some connected component c. Then there exists (a unique) $g : x_1 \to x_2$ with $F(g) \circ s_1 = s_2$. □

7 Ends and coends

7.1 Dinatural and extranatural transformations

8 Nerve and realization

Definition 8.1. The *simplex category* Δ has objects finite ordinals $[n] \overset{\text{df}}{=} \{0, \ldots, n\}$, $0 < n \in \omega$, and morphisms order-preserving maps between them.

Definition 8.2. A *simplicial object* $X_\bullet$ of a category $\mathbf{S}$ is a functor $X : \Delta^{\text{op}} \to \mathbf{S}$. We write X_n for $X([n])$ and $X_f : X_n \to X_m$ for the map $X(f)$ gotten by applying X to a map $f : [m] \to [n]$ in Δ.

A category $\mathbf{C}$ internal to $\mathbf{S}$ is a truncated simplicial object of $\mathbf{S}$. There is a universal simplicial object of $\mathbf{S}$ whose truncation to $n = 0$ and $n = 1$ is again $\mathbf{C}$. This is the nerve.

Definition 8.3. The *nerve* of an internal category $\mathbf{C}$ of $\mathbf{S}$ is the simplicial object $\mathsf{Nerve}(\mathbf{C})$ of $\mathbf{S}$, given by

$$\mathsf{Nerve}(\mathbf{C})_n \overset{\text{df}}{=} C_1 \times_{C_0} \times_{C_0} \ldots \ (n \text{ times}) \ \times_{C_0} C_1,$$

(so composable n-tuples of morphisms in $\mathbf{C}$), and for $f : [m] \to [n]$ in Δ,

$$(\mathsf{Nerve}(\mathbf{C})_f : \mathsf{Nerve}(\mathbf{C})_n \to \mathsf{Nerve}(\mathbf{C})_m) \overset{\text{df}}{=} \left(\begin{pmatrix} \phi_0 \\ \vdots \\ \phi_{n-1} \end{pmatrix} \mapsto \begin{pmatrix} \overline{\varphi}_0 \\ \vdots \\ \overline{\varphi}_{m-1} \end{pmatrix} \mapsto c \begin{pmatrix} \overline{\varphi}_0 \\ \vdots \\ \overline{\varphi}_{m-1} \end{pmatrix} = \begin{pmatrix} \psi_0 \\ \vdots \\ \psi_{m-1} \end{pmatrix} \right)$$

where $\overline{\varphi}_j$ is the n-tuple of maps (padded with identity maps as necessary) $\phi_{f(i+1)-1}, \ldots, \phi_{f(i)}$, and c is (repeated) composition applied to each of the rows.

Hence, if $\mathbf{S} = \mathbf{Set}$, $\mathsf{Nerve}(\mathbf{C})_f$ would be given by

$$\left(X_0 \overset{\phi_0}{\to} \dots \overset{\phi_{n-1}}{\to} X_n\right) \mapsto \left(X_{f(0)} \overset{\psi_0}{\to} \dots \overset{\psi_{m-1}}{\to} X_{f(m)}\right)$$

(the definition above works in any finitely complete category $\mathbf{S}$.)

Remark 8.4. In particular, the nerve of a definable category is a simplicial definable set.