# An introduction to abelian sheaf cohomology

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# 1 Basic definitions

## 1.1 Notation and conventions

- If $\mathbf{C}$ is a category, and $X$ and $Y$ are objects of $\mathbf{C}$, we will write
$$\mathbf{C}(X, Y)$$
for the set of morphisms $X \to Y$ in $\mathbf{C}$.
- If $X$ is a topological space, we will write $\mathscr{O}(X)$ for its poset of open subsets, viewed as a category.
- If $V$ is an open subset of $X$, by an **open covering** of $V$ we mean a family of open subsets $\left\{ U_i \subseteq V \,\middle|\, U_i \underset{\text{open}}{\subseteq} X \right\}_{i \in I}$ such that $\bigcup_{i \in I} U_i = V$.
- We will say that a poset is **filtered** if it is a directed set: for any finite number of elements $x_1, \ldots, x_n$ in the poset, there is an $x$ such that for all $i = 1, \ldots, n$, $x \geqslant x_i$. A poset which satisfies the dual condition will be called **cofiltered**.

- To us, "left-exact" and "right-exact" mean "preserves finite limits" and "preserves finite colimits", and "continuous" and "cocontinuous" mean "preserves small limits" and "preserves small colimits". When we are working in the context of functors between abelian categories, "left-exact" and "right-exact" will also stipulate *additivity*.

## 1.2 Sheaves and presheaves on a topological space

Let $X$ be a topological space.

**Definition 1.1.** A **presheaf** $P$ on $X$ is a contravariant functor

$$P : \mathscr{O}(X)^{\mathrm{op}} \to \mathbf{Set}.$$

**Example 1.2.** Any contravariant hom-functor $\mathscr{O}(X)(-, U)$ is a presheaf on $X$. It takes the value $1 = \{\varnothing\}$ (meaning "containment") on open subsets $V \subseteq U$ and the value $0 = \varnothing$ on open subsets $V \nsubseteq U$.

**Definition 1.3.** Every inclusion of open subsets $U \subseteq V$ is sent by a presheaf $P$ to a map $P(V) \to P(U)$. We call this map the *restriction map* from $P(V)$ to $P(U)$, and write it as $\mathrm{res}_{V,U}$.

**Definition 1.4.** The open neighborhoods of a single point $x \in X$ form a **neighborhood filter** $\{U \mid U \ni x\}$. These form a cofiltered diagram. Applying a presheaf $P$ to this cofiltered diagram yields (whence contravariance) a filtered diagram. The filtered colimit of this diagram

$$\varinjlim \left\{ P(U) \, \mathrm{res} \, P(U') \;\middle|\; U \supseteq U' \ni x \right\}$$

is called the **stalk** of $P$ at the point $x$, and is denoted $P_x$.

**Definition 1.5.** Let $P$ be a presheaf on $X$. Let $V$ be an open subset of $X$, and let $\{U_i\}_{i\in I}$ be an open covering of $V$. A **matching family** for $P$ with respect to this covering of $V$ is an $I$-indexed sequence

$$(s_i \in P(U_i))_{i \in I}$$

such that for all $U_i, U_j$ and $W \subseteq (U_i \cap U_j)$,

$$\mathrm{res}_{U_i \cap U_j, W}(s_i) = \mathrm{res}_{U_i \cap U_j, W}(s_j).$$

**Definition 1.6.** A **sheaf** $\mathcal{F}$ on $X$ is a presheaf on $X$ which satisfies the following additional condition: for every open subset $V \subseteq X$, for every open covering $\{U_i\}_{i\in I}$ of $V$, and for every matching family $(s_i)_{i\in I}$ for $P$ with respect to this covering of $V$, there exists a unique element $s \in P(V)$ "amalgamating" the matching family, i.e.

$$\forall i \in I \quad \mathrm{res}_{V,U_i}(s) = s_i. \tag{1}$$

**Proposition 1.7.** *An equivalent formulation of the sheaf condition 1 is: for every open subset $V$ of $X$, and for every open covering $\{U_i \to V\}_{i\in I}$ of $V$, the diagram*

$$\mathcal{F}(V) \to \prod_{i\in I} \mathcal{F}(U_i) \rightrightarrows \prod_{j,k\in I} \mathcal{F}(U_j \cap U_k) \tag{2}$$

*is an equalizer.*

*Proof.* Suppose first that $\mathcal{F}$ uniquely amalgamates matching families. Fix $V$ and an open covering $\{U_i\}_{i \in I}$ of $V$. We will exhibit the universal property of the equalizer diagram.

If $Y$ is a set which is a cocone to

$$\prod_{i \in I} \mathcal{F}(U_i) \rightrightarrows \prod_{j,k \in I} \mathcal{F}(U_j \cap U_k),$$

via

$$Y \xrightarrow{d} \prod_{i \in I} \mathcal{F}(U_i),$$

then we need to obtain a unique limiting map $c : Y \to \mathcal{F}(V)$.

Each element $y \in Y$ determines via $d$ a tuple $d(y) = (d(y)[i])_{i \in I}$ of elements, one from each $\mathcal{F}(U_i)$. Since $Y$ was a cocone, $d(y)$ is a matching family with a unique amalgamation $s_y$. So, we send $y \mapsto s_y$.

This is easily seen to be a cone map $Y \to \mathcal{F}(V)$. This map is unique because amalgamations are unique: if we had a different map $c'$, then it would follow that there is a matching family which two different amalgamations.

Conversely, suppose that $\mathcal{F}$ satisfies the sheaf condition in the sense of 2. A matching family $(s_i)_{i \in I}$ is a singleton cone $\{(s_i)_{i \in I}\}$ to the equalizer diagram. The uniqueness of the limiting map $\{(s_i)_{i \in I}\} \to \mathcal{F}(V)$ is then literally the uniqueness of the amalgamation. □

**Definition 1.8.** A **morphism** or **map** of presheaves $f : P_1 \to P_2$ is a natural transformation $P_1 \to P_2$ viewed as contravariant functors $\mathscr{O}(X)^{\mathrm{op}} \to \mathbf{Set}$.

**Example 1.9.** Let $E \xrightarrow{p} X$ be a local homeomorphism. Then assigning each $U \underset{\text{open}}{\subseteq} X$ to the set of **sections** (continuous functions $s : U \to E$ such that $p \circ s$ restricts to the identity map on $U$) with restriction maps literally restriction maps gives a sheaf $U \mapsto \Gamma(U, E)$, called the **sheaf of sections** of the local homeomorphism $E \xrightarrow{p} X$.

If $E \xrightarrow{p} X$ and $E' \xrightarrow{q} X$ are two local homeomorphisms and $f : E \to E'$ is a map of spaces over $X$, then composing by $f$ sends sections to sections and induces a morphism of sheaves $\Gamma(-, E) \xrightarrow{\overline{f}} \Gamma(-, E')$.

**Definition 1.10.** The presheaves on $X$ with presheaf morphisms between them form a category $\mathbf{PShv}(X)$. They contain as a full subcategory the category $\mathbf{Shv}(X)$ of sheaves on $X$.

There is a canonical way of turning presheaves into sheaves, by "freely adjoining unique amalgamations", called **sheafification**. One way of writing it down is to construct from every presheaf $P$ on $X$ a local homeomorphism $E(P) \xrightarrow{p} X$, where

$$\left( E(P) = \coprod_{x \in X} P_x \right) \twoheadrightarrow X$$

is topologized by pulling back the topology on $X$ along the obvious projection. The **sheafification** $\widetilde{P}$ of $P$ is then defined to be the sheaf of sections $\Gamma(-, E(P))$.

**Definition 1.11.** Let $\mathcal{F}$ be a sheaf on $X$. Viewing $X$ as an open subset of itself and borrowing the terminology from the above example, we say that the set $\mathcal{F}(X)$ is the set of **global sections** of $\mathcal{F}$, and we write

$$\Gamma(X, \mathcal{F}) \stackrel{\mathrm{df}}{=} \mathcal{F}(X).$$

Since global sections is an evaluation operation on a category of functors, it is (1) purely formal, so must apply to presheaves, and (2) is functorial by simply taking the component-at-$X$ of natural transformations: we therefore define the **global sections functor**

$$\Gamma(X, -) : \mathbf{PShv}(X) \to \mathbf{Set}$$

by

$$\left(P_1 \xrightarrow{\eta} P_2\right) \overset{\Gamma(X,-)}{\mapsto} \left(\Gamma(X, P_1) \xrightarrow{\eta_X} \Gamma(X, P_2)\right).$$

This is easily seen to restrict to a functor $\Gamma(X, -) : \mathbf{Shv}(X) \to \mathbf{Set}$, which we also call global sections.

**Proposition 1.12.** *Let* $\mathbf{1}$ *be the constant sheaf which sends an open subset* $U$ *to the terminal set* $1$. *There is a natural isomorphism of functors*

$$\mathbf{PShv}(X)(\mathbf{1}, -) \simeq \Gamma(X, -).$$

*Proof.* Exercise. □

**Remark 1.13.** Precomposing $\mathbf{PShv}(X)(\mathbf{1}, -)$ and $\Gamma(X, -) : \mathbf{PShv}(X) \to \mathbf{Set}$ by the forgetful functor $\mathbf{Shv}(X) \to \mathbf{PShv}(X)$ yields $\mathbf{Shv}(X)(\mathbf{1}, -)$ and $\Gamma(X, -) : \mathbf{Shv}(X) \to \mathbf{Set}$, so the proposition 1.12 implies that these two functors are naturally isomorphic too.

## 1.3 Sheaves of groups and sheaves of abelian groups

**Definition 1.14.** The **finite product theory of groups** is a category $T_{\mathbf{Grp}}$ with objects

$$1, G, G \times G, G \times G \times G, \ldots$$

corresponding to all finite products (including the empty product 1) of a distinguished object $G$. The maps include all canonical projections between the objects, and the following additional maps:

1. A map $e : 1 \to G$ ("identity"),
2. A map $(-)^{-1} : G \to G$ ("inverse"),
3. A map $m : G \times G \to G$ ("multiplication").

These maps make the following diagrams commute:

**(Multiplication is associative:)**

$$\begin{tikzcd} G \times G \times G \arrow[r, "\mathrm{id}_G \times M"] \arrow[d, "m \times \mathrm{id}_G"'] & G \times G \arrow[d, "m"] \\ G \times G \arrow[r, "m"'] & G \end{tikzcd}$$

**(Identity is an identity:)**

$$\begin{tikzcd} G \arrow[r, "(e, \mathrm{id}_G)"] \arrow[d, "(\mathrm{id}_G, e)"'] \arrow[dr, "\mathrm{id}_G"] & G \times G \arrow[d, "m"] \\ G \times G \arrow[r, "m"'] & G \end{tikzcd}$$

**(Inverse map is inverse map:)**

$$\begin{tikzcd} G \arrow[r, "\left(\mathrm{id}_G, (-)^{-1}\right)"] \arrow[d, "\left((-)^{-1}, \mathrm{id}_G\right)"'] \arrow[dr, "\mathrm{id}_G"] & G \times G \arrow[d, "m"] \\ G \times G \arrow[r, "m"'] & G \end{tikzcd}$$

**Definition 1.15.** The **finite product theory of abelian groups** is a category $T_{\mathbf{Ab}}$ which looks like $T_{\mathbf{Grp}}$ except the extra structure on $G$ must obey the obvious commutativity constraint. So, letting $\tau : G \times G \to G \times G$ be the "twist" map which interchanges coordinates (it is easy to obtain this diagramatically from the universal property of $G \times G$), we ask that the data $e, (-)^{-1}, m$ additionally satisfy that the diagram

**Multiplication is commutative:**

$$\begin{tikzcd} G \times G \arrow[r, "\tau"] \arrow[dr, "m"'] & G \times G \arrow[d, "m"] \\ & G \end{tikzcd}$$

commutes.

**Definition 1.16.** A **group object** in $\mathbf{Shv}(X)$ is a left-exact functor $T_{\mathbf{Grp}} \to \mathbf{Shv}(X)$. The distinguished object $G \in T_{\mathbf{Grp}}$ is sent by the functor to a sheaf $\mathbf{G}$ on $X$, which by abuse of notation we also call a group object.

An **abelian group object** in $\mathbf{Shv}(X)$ is, similarly, a left-exact functor $T_{\mathbf{Ab}} \to \mathbf{Shv}(X)$.

**Remark 1.17.** Since limits in $\mathbf{Shv}(X)$ are computed "pointwise", an (abelian) group object $\mathbf{G}$ in $\mathbf{Shv}(X)$ is the same thing as a sheaf $\mathbf{G} : \mathscr{O}(X)^{\mathrm{op}} \to \mathbf{Set}$ which factors into a sheaf landing in $\mathbf{Grp}$ (or $\mathbf{Ab}$) composed by the forgetful functor to $\mathbf{Set}$.

**Exercise 1.18.** Verify that $\mathbf{Set}$ is isomorphic to $\mathbf{Shv}(1)$, the category of sheaves on the terminal topological space 1. Verify that $\mathbf{Grp}$ is the category of group objects in $\mathbf{Shv}(1)$ and $\mathbf{Ab}$ is the category of abelian group objects in $\mathbf{Shv}(1)$.

**Exercise 1.19.** Write down the notion of a morphism of abelian group objects in a category with finite limits. (Hint: it should be a natural transformation of functors from $T_{\mathbf{Ab}}$!). Verify that this gives the correct definition of an abelian group homomorphism when the finitely complete category is taken to be **Set**. Verify that the category $\mathbf{Ab}(\mathbf{Shv}(X))$ of abelian group objects in $\mathbf{Shv}(X)$ with morphisms of abelian group objects between them forms an abelian category.

**Lemma 1.20.** *The global sections functor*

$$\Gamma(X,-) \simeq \mathbf{Shv}(X)(\mathbf{1},-) : \mathbf{Shv}(X) \to \mathbf{Set}$$

*is left-exact.*

*Proof.* This follows from the easily-checked fact that covariant hom-functors $\mathbf{C}(A,-)$ preserve small limits. $\square$

**Remark 1.21.** Since $\Gamma(X,-)$ is left-exact, composition sends abelian group objects $L : T_{\mathbf{Ab}} \to \mathbf{Shv}(X)$ to abelian group objects of **Set**, which are just abelian groups:

$$T_{\mathbf{Ab}} \xrightarrow{L} \mathbf{Shv}(X) \xrightarrow{\Gamma(X,-)} \mathbf{Set}.$$

Therefore, global sections restricts to a functor between categories of abelian group objects, and we have the following diagram:

$$\begin{tikzcd}
\mathbf{Ab}(\mathbf{Shv}(X)) \arrow[r, "\mathrm{forget}^{\mathbf{Ab}(\mathbf{Shv}(X))}_{\mathbf{Shv}(X)}"] \arrow[d, "\Gamma^{\mathbf{Ab}(\mathbf{Shv}(X))}_{\mathbf{Ab}}(X{,}-)"'] & \mathbf{Shv}(X) \arrow[d, "\Gamma^{\mathbf{Shv}(X)}_{\mathbf{Set}}(X{,}-)"] \\
\mathbf{Ab} \arrow[r, "\mathrm{forget}^{\mathbf{Ab}}_{\mathbf{Set}}"'] & \mathbf{Set}
\end{tikzcd}$$

**Proposition 1.22.** *The global sections functor*

$$\Gamma(X,-)^{\mathbf{Ab}(\mathbf{Shv}(X))}_{\mathbf{Ab}} : \mathbf{Ab}(\mathbf{Shv}(X)) \to \mathbf{Ab}$$

*preserves small limits, so it is left exact. It is also an additive functor between abelian categories.*

*Proof.* Let $\mathbf{D}$ be a diagram in $\mathbf{Ab}(\mathbf{Shv}(X))$. Since forget : $\mathbf{Ab} \to \mathbf{Set}$ is a right adjoint and preserves limits, and limits in $\mathbf{Ab}(\mathbf{Shv}(X))$ are computed "pointwise", the forgetful functor forget : $\mathbf{Ab}(\mathbf{Shv}(X)) \to \mathbf{Shv}(X)$ is left-exact. We also have that global sections $\mathbf{Shv}(X) \to \mathbf{Set}$ is left-exact. We therefore calculate:

$$\begin{aligned}
\mathrm{forget}^{\mathbf{Ab}}_{\mathbf{Set}}\, \Gamma^{\mathbf{Ab}(\mathbf{Shv}(X))}_{\mathbf{Ab}}(X, \varprojlim \mathbf{D}) &= \Gamma^{\mathbf{Shv}(X)}_{\mathbf{Set}}(X, \mathrm{forget}^{\mathbf{Ab}(\mathbf{Shv}(X))}_{\mathbf{Shv}(X)} \varprojlim \mathbf{D}) \\
&\simeq \Gamma^{\mathbf{Shv}(X)}_{\mathbf{Set}}(X, \varprojlim \mathrm{forget}^{\mathbf{Ab}(\mathbf{Shv}(X))}_{\mathbf{Shv}(X)} \mathbf{D}) \\
&\simeq \varprojlim \left(\Gamma^{\mathbf{Shv}(X)}_{\mathbf{Set}}(X, \mathrm{forget}^{\mathbf{Ab}(\mathbf{Shv}(X))}_{\mathbf{Shv}(X)} \mathbf{D})\right) \\
&= \varprojlim \left(\mathrm{forget}^{\mathbf{Ab}}_{\mathbf{Set}}\, \Gamma^{\mathbf{Ab}(\mathbf{Shv}(X))}_{\mathbf{Ab}}(X, \mathbf{D})\right) \\
&\simeq \mathrm{forget}^{\mathbf{Ab}}_{\mathbf{Set}} \varprojlim \left(\Gamma^{\mathbf{Ab}(\mathbf{Shv}(X))}_{\mathbf{Ab}}(X, \mathbf{D})\right).
\end{aligned}$$

Since the isomorphisms in the above calculation arise just from moving $\varprojlim$ around, it is easy to see that the composite isomorphism

$$\mathrm{forget}^{\mathbf{Ab}}_{\mathbf{Set}}\,\Gamma^{\mathbf{Ab}(\mathbf{Shv}(X))}_{\mathbf{Ab}}(X, \varprojlim \mathbf{D}) \simeq \mathrm{forget}^{\mathbf{Ab}}_{\mathbf{Set}} \varprojlim \left(\Gamma^{\mathbf{Ab}(\mathbf{Shv}(X))}_{\mathbf{Ab}}(X, \mathbf{D})\right)$$

is actually the identity map; it is a map of limit cones because limits in $\mathbf{Shv}(X)$ (and hence $\mathbf{Ab}(\mathbf{Shv}(X))$) are computed pointwise. Since the identity map is always a group homomorphism,

$$\Gamma^{\mathbf{Ab}(\mathbf{Shv}(X))}_{\mathbf{Ab}}(X, \varprojlim \mathbf{D}) \simeq \varprojlim \left(\Gamma^{\mathbf{Ab}(\mathbf{Shv}(X))}_{\mathbf{Ab}}(X, \mathbf{D})\right).$$

That taking global sections is an additive functor boils down to the fact that the $\mathbf{Ab}$-enrichment of $\mathbf{Ab}(\mathbf{Shv}(X))$ is done "pointwise": when showing $\mathbf{Ab}(\mathbf{Shv}(X))$ is abelian, one defines the sum $\eta + \rho$ of two abelian sheaf homomorphisms $\eta, \rho : \mathcal{A} \to \mathcal{B}$ as simply the componentwise sum of the abelian group homomorphisms $\eta_U, \rho_U : \mathcal{A}(U) \to \mathcal{B}(B)$. □

# 2 Sheaf cohomology via injective resolutions

To define sheaf cohomology, we will be using injective resolutions. First, we will show that these are in ready supply.

## 2.1 $\mathbf{Ab}(\mathrm{Sh}(X))$ has enough injectives

**Definition 2.1.** Let $\mathbf{C}$ be a category, and let $X$ be an object of $\mathbf{C}$. We say that $X$ is an **injective object** of $\mathbf{C}$ if for any monomorphism $Y \xrightarrow{i} Z$, and a map $Y \xrightarrow{f} X$, there exists some extension $g$ of $f$ along $i$ such that the diagram

$$\begin{tikzcd} Y \arrow[r, hook, "i"] \arrow[d, "f"'] & Z \arrow[dl, dashed, "g"] \\ X & \end{tikzcd}$$

commutes.

**Definition 2.2.** We say that a category $\mathbf{C}$ **has enough injectives** if for every object $A$, there exists a monomorphism $A \to X$ with $X$ injective.

**Theorem 2.3.** $\mathbf{Ab}(\mathbf{Shv}(X))$ *has enough injectives.*

*Proof.* We will take as given the fact that $\mathbf{Ab}$ has enough injectives.

Let $\mathcal{F}$ be a sheaf of abelian groups on $X$. For each $x \in X$, consider the stalk $\mathcal{F}_x$ and an embedding of $\mathcal{F}_x$ into an injective abelian group $\mathcal{I}^x$. Form the **skyscraper sheaf**

$$\widetilde{\mathcal{I}^x} \overset{\mathrm{df}}{=} U \mapsto \begin{cases} \mathcal{I}^x \text{ if } x \in U \\ 0 \text{ if } x \notin U. \end{cases}$$

where 0 is the trivial abelian group; we set $\mathrm{res}_{V,U}$ to be the terminal map to 0 if $V$ contains $x$ but $U$ doesn't; if $U$ contains $x$ then so does $V$, in which case $\mathrm{res}_{V,U}$ is $\mathrm{id}_{\mathcal{I}^x}$; and finally, if neither $V$ nor $U$ contains $x$, then $\mathrm{res}_{V,U}$ is just $\mathrm{id}_0$.

**Claim.** $\mathbf{Ab}(\mathbf{Shv}(X))(\mathcal{F}, \widetilde{\mathcal{I}^x}) \simeq \mathbf{Ab}(\mathcal{F}_x, \mathcal{I}^x)$.

*Proof of claim.* Any natural transformation $\eta : \mathcal{F} \to \widetilde{\mathcal{I}^x}$ turns $\mathcal{I}^x$ into a cocone over the filtered diagram $\{\mathcal{F}(U) \,|\, U \ni x\}$ and thus induces a map $\eta_x : \mathcal{F}_x \to \mathcal{I}^x$.

We see that $\eta \mapsto \eta_x$ is injective: if $\eta$ and $\eta'$ satisfy that $\eta_x = \eta'_x$, then computing $\mathcal{F}_x$ as the quotient

$$\mathcal{F}_x = \bigsqcup_{U \ni x} \mathcal{F}(U) \Big/ \big(u \simeq u' \iff \text{they eventually restrict to the same element}\big)$$

means that each component $\eta_U$ and $\eta'_U$ are constant on the equivalence classes induced by this quotient, and agree on each equivalence class. Since the equivalence classes in particular partition each $\mathcal{F}(U)$, $\eta = \eta'$.

We see also that the map $\eta \mapsto \eta_x$ is surjective: given a map $c : \mathcal{F}_x \to \mathcal{I}^x$, we lift it by decreeing that for $u \in \mathcal{F}(U)$, $\eta_U(u) \overset{\text{df}}{=} c([u])$, where $[u]$ is the germ of $[u]$. This is easily checked to be componentwise a map of abelian groups, and is therefore an element of $\mathbf{Ab}(\mathbf{Shv}(X))(\mathcal{F}, \widetilde{\mathcal{I}^x})$. $\square$

The claim implies that $\widetilde{\mathcal{I}^x}$ is an injective sheaf: given an injectivity diagram, we can just pass to the stalk at $x$ and then obtain an extension by lifting the extension between stalks along the isomorphism we just constructed, viz.

$$\begin{tikzcd} \mathcal{F} \arrow[r, "i"] \arrow[d, "f"'] & \mathcal{G} \arrow[dl, dashed, "\eta"] \\ \widetilde{\mathcal{I}^x} & \end{tikzcd} \quad \leftrightarrow \quad \begin{tikzcd} \mathcal{F}_x \arrow[r, "i_x"] \arrow[d, "f_x"'] & \mathcal{G}_x \arrow[dl, dashed, "\eta_x"] \\ \mathcal{I}^x. & \end{tikzcd}$$

One problem remains, however: while there is always some map $\eta : \mathcal{F} \to \widetilde{\mathcal{I}^x}$ (corresponding to the injection $\eta_x : \mathcal{F} \hookrightarrow \mathcal{I}^x$, whence our assumption when we obtained $\mathcal{I}^x$), it is not necessarily a monomorphism of sheaves. This is because outside of the neighborhood filter of $x$, the components of $\eta$ collapse to terminal maps.

The solution to this problem will be the same as when one exhibits that there are enough injective abelian groups: we *define*

$$\mathcal{I} \overset{\text{df}}{=} \prod_{x \in X} \widetilde{\mathcal{I}^x}$$

(remember that limits of sheaves are computed pointwise!). It is easy to see that an arbitrary product of injective objects is injective, and easy also to see that the corresponding product map

$$\left( \prod_{x \in X} \eta_x \right) : \mathcal{F} \to \mathcal{I}$$

is injective at every component. Hence, $\mathbf{Ab}(\mathbf{Shv}(X))$ has enough injectives. $\square$

## 2.2 Defining sheaf cohomology

Now, we will define sheaf cohomology for sheaves of abelian groups on $X$.

**Definition 2.4.** Let $\mathcal{F}$ be a sheaf of abelian groups on $X$. An **injective resolution** of $\mathcal{F}$ is a sequence of embeddings

$$0 \to \mathcal{F} \to \mathcal{I}_0 \to \mathcal{I}_1 \to \cdots .$$

Since $\mathbf{Ab}(\mathbf{Shv}(X))$ has enough injectives, an injective resolution may be constructed for any sheaf of abelian groups $\mathcal{F}$ by iteratively applying the construction from the proof of the theorem 2.3 to cokernels.

**Definition 2.5.** Let $\mathcal{F}$ be a sheaf of abelian groups. Let $n \geqslant 0$. Fix an injective resolution $0 \to \mathcal{F} \to \mathcal{I}_0 \to \mathcal{I}_1 \to \cdots$ of $\mathcal{F}$. Apply the global sections functor $\Gamma(X, -)$ to this resolution and chop off the first term to obtain the complex

$$0 \to \Gamma(X, \mathcal{I}_0) \xrightarrow{\delta^0} \Gamma(X, \mathcal{I}_1) \xrightarrow{\delta^1} \Gamma(X, \mathcal{I}_2) \to \cdots .$$

**The $n^{\mathbf{th}}$ cohomology group** $H^n(X, \mathcal{F})$ of $\mathcal{F}$ (or alternately, *of $X$ with values in $\mathcal{F}$*) is defined to be the homology of this complex:

$$H^n(X, \mathcal{F}) \overset{\mathrm{df}}{=} \ker(\delta^n) \Big/ \mathrm{im}(\delta^{n-1}).$$

**Theorem 2.6.** *The previous definition 2.5 is independent of the choice of injective resolution.*

*Proof.* We will show that, given two injective resolutions $(I_i)_{i\in\omega}$ and $(J_i)_{i\in\omega}$ of $\mathcal{F}$, then there exists a chain homotopy equivalence between the complexes

$$0 \to \Gamma(X, \mathcal{I}_0) \to \Gamma(X, \mathcal{I}_1) \to \Gamma(X, \mathcal{I}_2) \to \cdots$$

and

$$0 \to \Gamma(X, \mathcal{J}_0) \to \Gamma(X, \mathcal{J}_1) \to \Gamma(X, \mathcal{J}_2) \to \cdots ,$$

and chain homotopy equivalences induce isomorphisms in chain homology, which is what we want.

Since the global sections functor $\Gamma(X, -) : \mathbf{Ab}(\mathbf{Shv}(X)) \to \mathbf{Ab}$ is additive, it preserves the chain homotopy condition, so it suffices to exhibit a chain homotopy between the truncated complexes

$$(0 \to \mathcal{I}_0 \to \mathcal{I}_1 \to \cdots) \simeq (0 \to \mathcal{J}_0 \to \mathcal{J}_1 \to \cdots).$$

We inductively obtain a sequence of maps $\Phi_k : \mathcal{J}_k \to \mathcal{I}_k$, as in the diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{F} \arrow[r] & \mathcal{I}_0 \arrow[r] & \mathcal{I}_1 \arrow[r] & \cdots \\
0 \arrow[r] & \mathcal{F} \arrow[r] \arrow[u, "\mathrm{id}_{\mathcal{F}}"] & \mathcal{J}_0 \arrow[r] \arrow[u, dashed, "\Phi_0"] & \mathcal{J}_1 \arrow[r] \arrow[u, dashed, "\Phi_1"'] & \cdots
\end{tikzcd}$$

as follows: $\Phi_0$ is the extension of the composite map (going from bottom left, then up, then right) $\mathcal{F} \to \mathcal{I}_0$ along the embedding $\mathcal{F} \hookrightarrow \mathcal{J}_0$, and generally one obtains $\Phi_k : \mathcal{J}_k \to \mathcal{I}_k$ by checking that $\Phi_k$ extends in a well-defined way to a map from the cokernel $\operatorname{coker}(\mathcal{J}_{k-1} \to \mathcal{J}_k)$ to $\mathcal{J}_{k+1}$. Then one inductively applies injectivity.

Arguing symmetrically, we obtain a chain map $(\Psi_k : \mathcal{I}_k \to \mathcal{J}_k)_{k\in\omega}$.

We will show how to construct the chain homotopy $\Phi \circ \Psi \simeq \mathrm{id}_{\mathcal{I}}$; the rest of the argument will follow from symmetry.

It remains to exhibit chain homotopy maps $h^n$, as in the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{I}_0 \arrow[r, "\delta^0"] \arrow[d, shift left] \arrow[d, shift right] \arrow[dl, dashed, "h^0"'] & \mathcal{I}_1 \arrow[r, "\delta^1"] \arrow[d, shift left] \arrow[d, shift right] \arrow[dl, dashed, "h^1"'] & \cdots \arrow[dl, dashed, "h^2"'] \\
0 \arrow[r] & \mathcal{I}_0 \arrow[r, "\delta^0"] & \mathcal{I}_1 \arrow[r, "\delta^1"] & \cdots
\end{tikzcd}$$

(where the vertical maps at each $\mathcal{I}_k$ are meant to be $(\Phi \circ \Psi)_k$ and $\mathrm{id}_{\mathcal{I}_k}$), satisfying the *chain homotopy condition*:

$$\forall n \in \omega, \quad (\Phi \circ \Psi)_n - \mathrm{id}_n = \delta^{n-1} \circ h^n + h^{n+1} \circ \delta^n.$$

One obvious way of obtaining candidate $h^n$'s is by using injectivity. $h^n$ should be an extension of (something) along $\delta^{n-1}$ (generally the cokernel of $\delta^{n-2}$), so we should have

$$(\text{something}) = h^1 \circ \delta^{n-1}.$$

Looking at the chain homotopy condition, it is clear that we should define $h^{n+1}$ as the extension (gotten by injectivity) along $\delta^n$ of the map:

$$(\Phi \circ \Psi)_n - \mathrm{id}_n - \delta^{n-1} \circ h^n.$$

And $h^0$ is zero, which provides the base of the induction. □

# 3 Acyclic resolutions

**Definition 3.1.** We say that a sheaf $\mathcal{A}$ is **acyclic** if its cohomology vanishes in degree $\geqslant 1$.

**Definition 3.2.** An **acyclic resolution** of $\mathcal{F}$ is a long exact sequence

$$0 \to \mathcal{F} \to \mathcal{A}_0 \to \mathcal{A}_1 \to \mathcal{A}_2 \cdots$$

such that for all $n \in \omega$, $\mathcal{A}_n$ is acyclic.

**Theorem 3.3.** *Let*

$$0 \to \mathcal{F} \to \mathcal{A}_0 \xrightarrow{d^0} \mathcal{A}_1 \xrightarrow{d^1} \cdots$$

*be an acyclic resolution of $\mathcal{F}$. Then (with the convention that $d^{-1} = 0$, and writing $\delta^k = \Gamma(X, d^k)$,) there are natural isomorphisms*

$$\forall n, \quad H^n(X, \mathcal{F}) \simeq {}^{\ker \delta^n} / {}_{\operatorname{im} \delta^{n-1}}.$$

*This means that sheaf cohomology can be computed with acyclic resolutions.*

*Proof.* Recall that any long exact sequence, such as

$$0 \to \mathcal{F} \xrightarrow{d^{-1}} \mathcal{A}_0 \xrightarrow{d^0} \mathcal{A}_1 \xrightarrow{d^1} \cdots$$

is part of a diagram of interlocking short exact sequences, given by taking cokernels of the map $d^l$: writing $\mathcal{C}_k \overset{\text{df}}{=} \operatorname{coker}(d^k)$, we have the following diagram

$$\begin{tikzcd}
 & & 0 \arrow[dr] & & 0 & & 0 \arrow[dr] & & 0 \\
 & & & \mathcal{F} \arrow[ur] \arrow[dr] & & & & \mathcal{C}_0 \arrow[ur] \arrow[dr] & \\
0 \arrow[rr] & & \mathcal{F} \arrow[rr] \arrow[ur] & & \mathcal{A}_0 \arrow[rr] \arrow[dr] & & \mathcal{A}_1 \arrow[rr] \arrow[ur] & & \mathcal{A}_2 \cdots \\
 & 0 \arrow[ur] & & & & \mathcal{C}_{-1} \arrow[ur] \arrow[dr] & & & \\
 & & & & 0 \arrow[ur] & & 0 & &
\end{tikzcd}$$

where the diagonal sequences are short exact.

We will invoke the following important fact, which we leave as an exercise to the conscientious reader (proofs of this, and of the important "horseshoe lemma", may be found in [3] or [8].)

**Fact 3.4.** *A short exact sequence* $0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C}$ *in* $\mathbf{Ab}(\mathbf{Shv}(X))$ *induces a long exact sequence*

$$0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \xrightarrow{\partial^{-1}} H^0(X, \mathcal{A}) \to H^0(X, \mathcal{B}) \to H^0(X, \mathcal{C}) \xrightarrow{\partial^0} H^1(X, \mathcal{A}) \to \cdots$$

Writing out the long exact sequence in cohomology associated to the short exact sequence

$$0 \to \mathcal{F} \to \mathcal{A}_0 \to \mathcal{C}_{-1} \to 0,$$

and remembering that the $\mathcal{A}_k$ are acyclic, we obtain

$$\cdots \mathcal{C}_{-1} \xrightarrow{\partial^{-1}} H^0(X, \mathcal{F}) \to H^0(X, \mathcal{A}_0) \to H^0(X, \mathcal{C}_{-1}) \xrightarrow{\partial^0} H^1(X, \mathcal{F}) \to 0 \to H^1(X, \mathcal{C}_{-1}) \to \cdots$$
$$\ldots H^{n-1}(X, \mathcal{F}) \to 0 \to H^{n-1}(X, \mathcal{C}_{-1}) \xrightarrow{\partial^{n-1}} H^n(X, \mathcal{F}) \to 0 \to H^n(X, \mathcal{C}_{-1}) \xrightarrow{\partial^n} \cdots$$

By exactness of the sequence, when $n = 1$, we have

$$H^1(X, \mathcal{F}) \simeq \operatorname{coker}\left(\Gamma(X, \mathcal{A}_0) \to \Gamma(X, \mathcal{C}_{-1})\right),$$

and for general $n$,

$$H^n(X, \mathcal{F}) \simeq H^{n-1}(X, \mathcal{C}_{-1})$$

via the connecting homomorphism $\partial^{n-1}$.

Now, what has happened here is that $H^n$ of $\mathcal{F}$, which appears at the beginning of the short exact sequence $0 \to \mathcal{F} \to \mathcal{A}_0 \to \mathcal{C}_{-1}$, is isomorphic by the connecting homomorphism $\partial^{n-1}$ to $H^{n-1}$ of $\mathcal{C}_{-1}$, which appears at the end of the short exact sequence.

Since each $\mathcal{C}_k$ shows up in two short exact sequences in the short exact sequence decomposition of the long exact sequence $0 \to \mathcal{F} \to \mathcal{A}_0 \to \dots$, once at the end of a sequence and then at the beginning of the next sequence, we can iterate the above argument. Explicitly, let us additionally write $\partial^{n-1}_{\mathcal{A}_k}$ for $(n-1)^{\text{th}}$ connecting homomorphism in the long exact sequence associated to the short exact sequence $0 \to \mathcal{C}_{k-2} \to \mathcal{A}_k \to \mathcal{C}_{k-1} \to 0$. Then we obtain a chain of isomorphisms

$$H^n(X,\mathcal{F}) \overset{\partial^{n-1}_{\mathcal{A}_0}}{\simeq} H^{n-1}(X,\mathcal{C}_{-1}) \overset{\partial^{n-2}_{\mathcal{A}_1}}{\simeq} H^{n-2}(X,\mathcal{C}_0) \overset{\partial^{n-3}_{\mathcal{A}_2}}{\simeq} \dots \overset{\partial^{1}_{\mathcal{A}_{n-2}}}{\simeq} H^1(X,\mathcal{C}_{n-3}).$$

Arguing as we did for $H^1(X,\mathcal{F})$, we see that

$$H^1(X,\mathcal{C}_{n-3}) \simeq \operatorname{coker}\left(\Gamma(X,\mathcal{A}_{n-1}) \to \Gamma(X,\mathcal{C}_{n-2})\right).$$

Since the short exact sequences $0 \to \mathcal{C}_{n-2} \to \mathcal{A}_n \to \mathcal{C}_{n-1} \to 0$ interlocked, $C_{n-2} \simeq \ker\left(A_n \to C_{n-1}\right)$; since $\Gamma(X,-)$ is left-exact, it commutes with kernels, so

$$\Gamma(X,\mathcal{C}_{n-2}) \simeq \ker\left(\Gamma(X,\mathcal{A}_n) \to \Gamma(X,\mathcal{C}_{n-1})\right).$$

Now it suffices to show that $\ker\left(\Gamma(X,\mathcal{A}_n) \to \Gamma(X,\mathcal{C}_{n-1})\right)$ is isomorphic to $\ker(\delta^n)$ and that the image of

$$\Gamma(X,\mathcal{A}_{n-1}) \to \Gamma(X,\mathcal{C}_{n-2})$$

is isomorphic to $\operatorname{im}(\delta^{n-1})$; this will give an isomorphism $H^n(X,\mathcal{F})$ with the homology of the complex $0 \to \Gamma(X,\mathcal{A}_0) \to \cdots$ when $n > 0$ (the case $n = 0$ is trivial).

We calculate:

$$\begin{aligned}
\ker(\delta^n) &= \ker(\Gamma(X,d^n)) \\
&\simeq \Gamma(X,\ker(d^n)) \\
&\simeq \Gamma\left(\ker\left(\mathcal{A}_n \to \mathcal{C}_{n-1}\right)\right) \\
&\simeq \ker\left(\Gamma(X,\mathcal{A}_n) \xrightarrow{\delta^n} \Gamma(X,\mathcal{C}_{n-1})\right),
\end{aligned}$$

and since $\mathcal{C}_{n-2} \to \mathcal{A}_n$ was mono, then by the left-exactness of $\Gamma$, $\Gamma(X,\mathcal{C}_{n-2}) \to \Gamma(X,\mathcal{A}_n)$ is an injective map of abelian groups. Therefore,

$$\begin{aligned}
\operatorname{im}\left(\Gamma(X,\mathcal{A}_{n-1}) \to \Gamma(X,\mathcal{C}_{n-2})\right) &\simeq \operatorname{im}\left(\Gamma(X,\mathcal{A}_{n-1}) \to \Gamma(X,\mathcal{C}_{n-2}) \to \Gamma(X,\mathcal{A}_n)\right) \\
&\simeq \operatorname{im}\left(\Gamma(X,\mathcal{A}_{n-1}) \to \Gamma(X,\mathcal{A}_n)\right) \\
&\simeq \operatorname{im}\left(\delta^{n-1}\right),
\end{aligned}$$

and the proof is complete. $\square$

We conclude with an exercise:

**Exercise 3.5.** Work out the notion of a ring object in $\mathbf{Shv}(X)$ and that show that it is the same thing as a sheaf of rings on $X$. Given an ring object $\mathcal{R}$ in $\mathbf{Shv}(X)$, work out the notion of an $\mathcal{R}$-module object in $\mathbf{Shv}(X)$.

Verify that the category of $\mathcal{R}$-module objects in $\mathbf{Shv}(X)$ (which we'll just call $\mathcal{R}$-modules) is an abelian category, and that everything we have done in this document generalizes to a notion of sheaf cohomology for sheaves of $\mathcal{R}$-modules on $X$. (Usually the notation for a sheaf of rings on $X$ is $\mathcal{O}_X$, and these things are usually called $\mathcal{O}_X$-modules.)

Sanity check: what is the ring object $\mathcal{R}$ which realizes $\mathbf{Ab}(\mathbf{Shv}(X))$ as the category of $\mathcal{R}$-modules on $X$?

Dually, prove that there exists a forgetful functor from the category of $\mathcal{R}$-modules to $\mathbf{Ab}(\mathbf{Shv}(X))$, simply by forgetting the $\mathcal{R}$-action on every $U \underset{\text{open}}{\subseteq} X$.

*Finale*: now open the nearest copy of Gelfand and Manin's *Methods of Homological Algebra* [5], turn to III.8, behold Theorem 3, and despair at all the work you have done: computing the sheaf cohomology of an $\mathcal{R}$-module yields the same result as computing the sheaf cohomology of the underlying sheaf of abelian groups.

# References

[1] M. Barr and C. Wells, *Toposes, theories, and triples*, TAC Reprints, 2005.

[2] G. Bredon, *Sheaf Theory*, Graduate Texts in Mathematics, Springer New York, 1997.

[3] H. Cartan and S. Eilenberg, *Homological Algebra (PMS-19)*, vol. 19, Princeton University Press, 2016.

[4] P. Garrett, *Sheaf cohomology*, notes on webpage, (2005).

[5] S. I. Gelfand and Y. I. Manin, *Methods of homological algebra*, Springer Science & Business Media, 2013.

[6] S. MacLane and I. Moerdijk, *Sheaves in Geometry and Logic: A First Introduction to Topos Theory*, Universitext, Springer New York, 1994.

[7] J.-P. Serre, *Faisceaux algebriques coherents*, The Annals of Mathematics, 61 (1955), pp. 197–278.

[8] C. A. Weibel, *An introduction to homological algebra*, no. 38, Cambridge university press, 1995.